Ideals of Completely Bounded Operators

Stephen David Allen

Presented for the degree of Doctor of Philosophy

University of Edinburgh, 1991

Acknowledgements

I would like to thank my supervisor, Dr A. M. Sinclair for his constant support and help. I would also like to thank Dr T. A. Gillespie and Dr A. G. Robertson for useful discussions.

I am grateful to Professor R. R. Smith for the opportunity to visit Texas A & M University and for invaluable discussions there.

Finally, little of this work would have been possible without the access to preprints and unpublished manuscripts that I enjoyed. In particular I would like to thank Professors V. I. Paulsen, D. P. Blecher, E. G. Effros, Z-J. Ruan, and F. Lust-Piquard for allowing me to use their results.

Abstract

The ideal structures of the Haagerup and weak*-Haagerup tensor products of $B(H)$ with itself are explored: the closed ideals of $B(H) \otimes_h B(H)$ are completely determined and some of the closed ideals of $B(H) \otimes_{w^*h} B(H)$ are identified. It is shown that if A is the Calkin algebra then $A \otimes_h A$ is simple. An example of a compact completely bounded operator on $K(H)$ which is not approximable in completely bounded operator norm by finite rank operators is constructed.

It is shown that a completely bounded operator φ on a C*-algebra A is p-summing for $1 \leqslant p \leqslant 2$ if it has a representation $\varphi(x) = V^*\rho(x)W$, $x \in A$ with $V, W \in \mathcal{C}_p$. It is also shown that if a completely bounded operator φ has a representation $\varphi = V^*xW$ and is p-summing for $p \geqslant 2$ then we can choose $V, W \in \mathcal{C}_p$.

It is shown that a version of the Grothendieck inequality using the usual non-symmetric modulus holds for completely bounded operators on a 2-concave subspace of a C*-algebra, and that if such an inequality is satisfied by all completely bounded operators on a subspace of a C*-algebra then that subspace is 2-concave.

Contents

Introduction

Completely bounded operators have an abstract metrical definition in terms of bounding the norm of arbitrarily large matrices. This can seem intimidating as few people relish the thought of calculating matrix norms. However we are very fortunate in having an algebraic representation of completely bounded operators, proved independently by Haagerup, Paulsen and Wittstock. This is usually much easier to work with, essentially saying that any completely bounded map from a C*-algebra A into $B(H)$ is composed of a representation π of A on a Hilbert space K and multiplication on the left and right by bounded operators V and W. Since these operations are both well understood, this characterisation is most useful; unfortunately the representation is not unique.

The chief drawback of this representation is that it does not automatically contain any further information that may be known about the operator. For example, if the operator $\varphi(x) = V^*\pi(x)W$ is known to be compact, what can be said about V and W? In particular, can they be chosen to be compact? It is questions of this nature that we attempt to address here.

One of the first problems that arises is the representation π. In general we know very little about how $\pi(A)$ sits inside $B(K)$ and this can make calculations in $B(K)$ difficult. We can circumvent this problem by restricting our attention to normal completely bounded maps from $B(H)$ to $B(H)$, or, equivalently, completely bounded maps from $K(H)$ to $B(H)$, for Haagerup has neatly characterised such maps as being sums of left and right multiplications, the representation being just an amplification, as all irreducible representations of $K(H)$ are unitarily equivalent to the identity.

Compact completely positive operators have been used by Choda [C]. She considers factors of type II_1 with trace τ which she says are of Haagerup type if there exists a net (P_α) of normal linear maps which are compact, completely positive and have the property that

$$\|P_\alpha(x) - x\|_2 \to 0, \qquad x \in N,$$

where $\|x\|_2 = \tau(x^*x)^{\frac{1}{2}}$. She then gives conditions on a group G for the group von Neumann algebra $vN(G)$ to be of Haagerup type.

It should be noted that not all bounded operators are completely bounded: the canonical example of a map that is bounded but not completely bounded is the transpose map on $\oplus_1^\infty M_n(\mathbb{C})$. The set of completely bounded maps on a C*-algebra is a Banach algebra and is to some extent better behaved than the algebra of bounded operators, not least because we have a representation theorem for completely bounded operators which we do not have for all bounded operators. For example, $B(H)$ does not have the approximation property, so compact operators on $B(H)$ are not in general approximable in norm by finite ranks: this means that $B(B(H))$ has a complicated structure. One might hope that completely bounded compact operators on $B(H)$ are approximable in completely bounded norm by finite ranks, and thus demonstrate that $CB(B(H))$ has a simpler structure. Unfortunately this turns out not to be the case as we shall see in Chapter 3.

The study of completely bounded operators is now well advanced; for an account of the history the notes in [Pa] at the end of each chapter are very good. The development has recently been rapid, often with the same results obtained or similar techniques used by different authors. Results which are of interest to the non-specialist have been found using these techniques: for example studying the Hochschild cohomology of von Neumann algebras [CS3, CES] and Kadison's similarity problem. Kadison asked in [K] when a non-selfadjoint representation π of a C*-algebra on a Hilbert space H is similar to a *-representation; that is when does there exist an invertible operator T on H such that $T\pi(\cdot)T^{-1}$ is a *-representation (in C*-algebra theory representations are generally assumed to be *-representations and we will not usually highlight the distinction). The question was motivated by the corresponding one for groups which has a negative answer. Haagerup [H1] proved that a bounded non-degenerate representation of a C*-algebra A on a Hilbert space H is similar to a *-representation if and only if it is completely bounded, thereby partially answering Kadison's question.

The introduction of the Haagerup tensor product as it is now called in [H4] related the study of completely bounded operators to that of tensor products. There are natural parallels of work by Grothendieck and others, which has been undertaken by Blecher, Paulsen, Effros and Ruan [BP, B1, B2, B3, ER1, ER2, ER3]. Recently Blecher and Smith [BS] have identified $CB(K(H), B(H))$ as $B(H) \otimes_{w^*h} B(H)$, the weak*-

Haagerup tensor product which they define as the dual of the Haagerup tensor product $T(H) \otimes_h T(H)$, where $T(H)$ is the trace-class operators, considered as $CB(K(H), \mathbb{C})$ in order to define a matrix norm structure. With this identification, it is natural to ask about the ideal structure of $B(H) \otimes_{w^*h} B(H)$, which we explore in Chapter 3.

The work of these authors is conducted in the class of operator spaces, which can be considered as subspaces of a C*-algebra, which is then represented on a Hilbert space H. Then using the isomorphism

$$M_n(B(H)) \cong B(H^n)$$

we can define a norm on $M_n(X)$ where X is the operator space. There are natural functors from the category of Banach spaces and bounded maps to the category of operator spaces and completely bounded maps (see [BP]). We however do not work with operator spaces, and only state theorems using isomorphisms not complete isomorphisms. In view of this functor one can ask how other properties of operators carry over: for example is there an analogue of p-summing? Certainly we cannot just ask that $\varphi_n : M_n(A) \to M_n(A)$ be p-summing for all n since $\varphi_n = \varphi \otimes \iota_n$ and $\pi_2(\iota_n) = n$ [Pi4]. This would mean that $\pi_2(\varphi_n) = \sqrt{n}\pi_2(\varphi)$. Effros and Ruan [ER4] have recently defined an analogue of 2-summing, saying that a map φ between operator spaces V and W is 2-column summing if the map

$$\iota \otimes \varphi : T(H) \otimes_h V \to T(H) \otimes_{max} W$$

is completely bounded where ι is the identity map on the trace class operators $T(H) = CB(K(H), \mathbb{C})$ and $\otimes_{max}$ is the maximal operator space tensor product (there is also a definition of 2-row summing). They then show this is equivalent to factoring through a Hilbert column space. We however shall simply ask what we can say about the representation of a completely bounded map which is p-summing.

We now outline the contents of the thesis.

In chapter 1, we make some introductory definitions and notations which will be of use throughout the thesis. We briefly sketch the background of completely bounded operators and tensor products. We also include some well-known results on compact operators for reference and define conditional expectations.

In chapter 2, we define the Haagerup and weak*-Haagerup tensor products. We include results from papers of Smith, Blecher and Smith and others where these are as yet unpublished. We show that the set of completely bounded operators from $K(H)$ to $B(H)$ may be considered as the weak*-Haagerup tensor product $B(H) \otimes_{w^*h} B(H)$.

In chapter 3, we consider ideals of $B(H) \otimes_h B(H)$ and $B(H) \otimes_{w^*h} B(H)$. We identify completely the closed ideals of the the former and find some of the closed ideals of the latter. In so doing, we construct a completely bounded operator on $K(H)$ which is compact yet not approximable in completely bounded norm by finite ranks. This shows that the set of compact completely bounded operators from $K(H)$ to $B(H)$ and the completely bounded norm closure of the finite ranks form distinct ideals of $B(H) \otimes_{w^*h} B(H)$. It also demonstrates that if $\varphi(x) = V^*\pi(x)W$ is compact then we cannot in general choose V and W to be compact.

In chapter 4, we turn our attention to p-summing operators. We formulate a necessary condition for a completely bounded operator to be p-summing for $p \geqslant 2$ and a sufficient condition for it to be p-summing for $1 \leqslant p \leqslant 2$.

In chapter 5, we look at the Grothendieck-Pisier-Haagerup inequality and ask whether the symmetrised modulus used by Pisier can be removed if we assume the operator is completely bounded. This is not always so, but we define a condition which is sufficient and then demonstrate its necessity.

This work was completed under the supervision of A. M. Sinclair.

1 Definitions and Notation

In this chapter, we introduce some notation and cover background material which will be of use throughout the later chapters.

In section 1 we introduce the injective and projective Banach space tensor products and relate them to the approximation property.

In section 2 we briefly discuss the definitions of completely positive and completely bounded operators and mention the representation theorems for these operators. Paulsen's book [Pa] contains much more background on completely bounded operators, and detailed proofs of the theorems we quote here.

In section 3 we discuss the properties of compact operators that we will need later and prove that $K(H)$ is the only non-zero proper closed ideal of $B(H)$. We also mention the von Neumann-Schatten classes $\mathcal{C}_p$ and weakly compact operators. Finally, we reproduce a result from Rickart [R], that all irreducible representations of $K(H)$ are unitarily equivalent to the identity.

In section 4 we define conditional expectations.

Firstly a note about the notation used in the thesis. We denote by H a Hilbert space which we will always assume to be separable. Many of the results are also true for non-separable Hilbert spaces but it makes the notation easier if separability is assumed; however the results on ideals do need separability so it should not be assumed that all results will generalise. All theorems are stated in terms of maps from $B(H)$ to $B(H)$; there is no greater generality in stating results about operators from $B(H_1, H_2)$ to $B(H_3, H_4)$ for Hilbert spaces $H_j,\ j = 1, \ldots, 4$ as we could consider suitable maps from $B(H_1 \oplus H_2 \oplus H_3 \oplus H_4)$ to itself.

1.1 Tensors products and the approximation property

We will always denote the algebraic tensor product of two Banach spaces X and Y by $X \odot Y$. Each element $u \in X \odot Y$ has a representation

$$u = \sum_{j=1}^{n} x_j \otimes y_j, \qquad x_j \in X,\ y_j \in Y. \tag{1}$$

We can also consider u as a bilinear form on $X^* \times Y^*$ mapping the element $(x^*, y^*) \in X^* \times Y^*$ to $\sum x^*(x_j) y^*(y_j)$.

We can also associate with u a finite rank operator $\bar{u} : X^* \to Y$ by

$$\bar{u}(x^*) = \sum_j x^*(x_j)y_j, \qquad x^* \in X^*.$$

The injective tensor product norm $\|\cdot\|_\vee$ is defined on $X \odot Y$ by

$$\|u\|_\vee = \|\bar{u}\|_{B(X^*,Y)} = sup\left\{\left|\sum_j x^*(x_j)y^*(y_j)\right| : x^* \in X^*,\ \|x^*\| \leqslant 1,\ y^* \in Y^*,\ \|y^*\| \leqslant 1\right\}.$$

We denote by $X\check{\otimes}Y$ the completion of $X \odot Y$ in the injective norm.

The projective tensor product norm $\|\cdot\|_\wedge$ is defined on $X \odot Y$ by

$$\|u\|_\wedge = inf\left\{\sum_j \|x_j\| \cdot \|y_j\|\right\},$$

where the infimum runs over all possible representations (1). We denote by $X\hat{\otimes}Y$ the completion of $X \odot Y$ in the projective norm.

Clearly, we have

$$\|u\|_\vee \leqslant \|u\|_\wedge.$$

Thus there is a natural norm decreasing map $J : X\hat{\otimes}Y \to X\check{\otimes}Y$. In general J is not injective: injectivity of this map is related to the approximation property which has been the subject of much study.

Definition 1.1.1 *(i) Let X, Y be Banach spaces. An operator $\varphi : X \to Y$ is said to be approximable if it can be approximated uniformly on every compact subset of X by finite rank operators. In other words, for all $\varepsilon > 0$ and all compact subsets K of X, there is a finite rank operator $\psi : X \to Y$ such that*

$$sup\{\|\varphi(x) - \psi(x)\| : x \in K\} < \varepsilon.$$

(ii) Suppose that λ is a positive real number. We will say that φ is λ-approximable if there is a net of finite rank operators $\psi_j : X \to Y$, such that $\|\psi_j\| \leqslant \lambda$, which converges uniformly on the compact subsets of X to the operator φ.

(iii) We will say that a Banach space X has the approximation property if the identity operator on X is approximable and that X has the λ-bounded approximation property if the identity operator on X is λ-approximable. Finally, we say that X has the bounded approximation property if it has the λ-bounded approximation property for some λ.

The following characterisation of spaces with the approximation property is due to Grothendieck [G1].

Theorem 1.1.2 *Let X be a Banach space. Then the following are equivalent.*

(i) X has the approximation property.

(ii) The natural map $J : X^*\hat{\otimes}X \to X^*\check{\otimes}X$ is injective.

(iii) For every Banach space Y the finite rank operators are dense in $B(Y,X)$ in the topology of uniform convergence on compact sets.

(iv) For every Banach space Y the finite rank operators are dense in $B(X,Y)$ in the topology of uniform convergence on compact sets.

(v) For every choice of $\{x_n\}_{n=1}^{\infty} \subset X$, $\{x_n^*\}_{n=1}^{\infty} \subset X^*$ such that $\sum_{n=1}^{\infty} \|x_n^*\| \cdot \|x_n\| < \infty$ and $\sum_{n=1}^{\infty} x_n^*(x)x_n = 0$ for all $x \in X$ we have $\sum_{n=1}^{\infty} x_n^*(x_n) = 0$.

(vi) For every Banach space Y, every compact operator $T \in B(Y,X)$ and every $\varepsilon > 0$ there is a finite rank operator $T_1 \in B(Y,X)$ with $\|T - T_1\| < \varepsilon$.

It was for some time an open problem whether there existed Banach spaces which did not satisfy the approximation property, but Enflo [En] and later Davie [Da] showed that such spaces did exist. Szankowski [Sz] showed more recently that in fact if H is a Hilbert space then $B(H)$ does not have the approximation property.

For more on the approximation property and its relations to the study of tensor products see Lindenstrauss and Tzafriri Volume 1 §1.e [LT].

1.2 Completely bounded operators

The following brief introduction to completely bounded operators follows the Christensen and Sinclair survey paper [CS2] very closely. This together with Paulsen's book [Pa] provides an excellent introduction to the subject.

If A is a C*-algebra, then there is a natural way to define a norm on $M_n(A)$, the algebra of $n \times n$ matrices with entries from A. There exists a faithful *-representation of A on a Hilbert space H ([T] Section IV.4). Then $M_n(A)$ embeds in $B(H^n)$, giving an operator norm for $M_n(A)$, and in fact this norm is unique.

To see this note that $M_n(A)$ is isomorphic to $M_n(\mathbb{C}) \odot A$ by the correspondence

$$(a_{ij}) \longleftrightarrow \sum_{i,j=1}^{n} e_{ij} \otimes a_{ij}, \qquad a_{ij} \in A,$$

where $\{e_{ij}\}$ are the matrix units for $M_n(\mathbb{C})$. In general there is no unique norm on the tensor product of two C*-algebras, but if one of the C*-algebras is $M_n(\mathbb{C})$, then the norm is unique. If π is a representation of A on a Hilbert space H then $\pi \otimes \iota$ is a representation of $A \odot M_n(\mathbb{C})$ on $H \otimes \mathbb{C}^n$ given by

$$\pi \otimes \iota \left(\sum_{j=1}^{n} a_j \otimes b_j \right) = \sum_{j=1}^{n} \pi(a_j) \otimes b_j.$$

Then we can complete $M_n(\mathbb{C}) \odot A$ in the norm inherited from $B(\mathbb{C}^n \otimes H)$, giving a tensor norm, which is then unique.

Definition 1.2.1 *(i) Let A and B be C^*-algebras, and let $\varphi : A \to B$ be a linear operator. Define $\varphi_n = \varphi \otimes \iota_n$ from $A \otimes M_n(\mathbb{C})$ into $B \otimes M_n(\mathbb{C})$, where ι_n is the identity operator on $M_n(\mathbb{C})$; that is*

$$\varphi_n((a_{ij})) = (\varphi(a_{ij})), \qquad (a_{ij}) \in M_n(A).$$

(ii) An operator $\varphi : A \to B$ is positive if $\varphi(a) \geqslant 0$ for all $a \geqslant 0 \in A$ and φ is completely positive if $\varphi_n : M_n(A) \to M_n(B)$ is positive for all $n \in \mathbb{N}$. Recall that an element a in a C^*-algebra A is positive ($a \geqslant 0$) if and only if $a = b^*b$ for some $b \in A$.[T]

(iii) An operator φ is completely bounded if $\{\|\varphi_n\| : n \in \mathbb{N}\}$ is bounded, and the completely bounded norm $\|\cdot\|_{cb}$ is defined by

$$\|\varphi\|_{cb} = sup\{\|\varphi_n\| : n \in \mathbb{N}\}.$$

The space of completely bounded operators form A into B with this norm is denoted by $CB(A,B)$.

Suppose $\varphi, \psi \in CB(A,B)$. Then

$$\begin{aligned}
\|\varphi_n + \psi_n\| &\leqslant sup\{\|(\varphi_n(x_{ij}) + \psi_n(x_{ij}))\| : (x_{ij}) \in M_n(A), \|(x_{ij})\| \leqslant 1\} \\
&\leqslant sup\{\|(\varphi_n(x_{ij}))\| + \|\psi_n(x_{ij}))\| : (x_{ij}) \in M_n(A), \|(x_{ij})\| \leqslant 1\} \\
&\leq \|\varphi_n\| + \|\psi_n\|,
\end{aligned}$$

and hence $\|\varphi+\psi\|_{cb} \leqslant \|\varphi\|_{cb} + \|\psi\|_{cb}$. Thus it is easy to see that $CB(A,B)$ is a Banach space; also if $\varphi \in CB(A,B)$, $\psi \in CB(B,C)$, then

$$\begin{aligned}
\|(\psi \circ \varphi)_n\| &= sup\{\|(\psi \circ \varphi)_n(x_{ij})\| : (x_{ij}) \in M_n(A), \|(x_{ij})\| \leqslant 1\} \\
&= sup\{\|(\psi(\varphi(x_{ij})))\| : (x_{ij}) \in M_n(A), \|(x_{ij})\| \leqslant 1\} \\
&\leqslant \|\psi_n\| \; sup\{\|(\varphi(x_{ij}))\| : (x_{ij}) \in M_n(A), \|(x_{ij})\| \leqslant 1\} \\
&= \|\varphi_n\| \cdot \|\psi_n\|,
\end{aligned}$$

and hence $\varphi \circ \psi \in CB(A,C)$ with $\|\psi \circ \varphi\|_{cb} \leqslant \|\varphi\|_{cb} \cdot \|\psi\|_{cb}$.

Completely positive operators have been studied in much detail and usefully characterised by Stinespring, as a combination of a representation and a left and a right multiplication.

Theorem 1.2.2 ([St]) *Let φ be a linear operator from a C^*-algebra A into $B(H)$. Then φ is completely positive if and only if there is a Hilbert space K, a representation π of A on K, and a bounded linear operator $V : H \to K$ such that*

$$\varphi(x) = V^*\pi(x)V, \qquad x \in A.$$

Further, $\|V\|^2 = \|\varphi\|$, and $V\pi(A)H$ is dense in K. This representation of φ is unique up to unitary equivalence.

This is a generalisation of the Gelfand-Naimark-Segal construction for a positive linear functional, and the proof is analagous. We define a sesquilinear form $<,>$ on $A \otimes H$ by

$$< a \otimes x, b \otimes y >=< \varphi(b^*a)x, y >_H,$$

and extending linearly. This is positive semi-definite so we can quotient out by the kernel to get an inner product space and complete this to a Hilbert space K. Then for $a \in A$ define $\pi(a) : A \otimes H \to A \otimes H$ by

$$\pi(a)\sum_j a_j \otimes x_j = \sum_j aa_j \otimes x_j.$$

This leaves $Ker <,>$ invariant and is bounded so may be extended to a bounded operator on K. Then define $V : H \to K$ by $V(x) = 1 \otimes x + Ker <,>$ and this will give the required representation.

The uniqueness up to unitary equivalence says that if (K_1, π_1, V_1) and (K_2, π_2, V_2) are two representations of φ with the properties of the theorem, then there is a unitary operator $U : K_1 \to K_2$ with $UV_1 = V_2$ and

$$U\pi_1(x)U^* = \pi_2(x), \qquad x \in A.$$

There is a corresponding result for completely bounded operators due to Haagerup, Paulsen and Wittstock independently, but we do not get a unique representation.

Theorem 1.2.3 *Let φ be a linear operator from a C^*-algebra A into $B(H)$. Then φ is completely bounded if and only if there is a Hilbert space K, a representation π of A on K, and bounded linear operators $V, W : H \to K$ such that*

$$\varphi(x) = V^*\pi(x)W, \qquad x \in A.$$

Further, $\|\varphi\|_{cb} \leqslant \|V\| \cdot \|W\|$, and there is such a representation of φ with $\|\varphi\|_{cb} = \|V\| \cdot \|W\|$.

The proof uses the Arveson extension theorem [Ar] and the following theorem.

Theorem 1.2.4 ([Pa], Theorem 7.3) *Let A be a unital C^*-algebra and let $\varphi : A \to B(H)$ be a completely bounded map. Then there exist completely positive maps φ_1, $\varphi_2 : A \to B(H)$ with $\|\varphi_1\|_{cb} = \|\varphi_2\|_{cb} = \|\varphi\|_{cb}$ such that the map $\Phi : M_2(A) \to B(H \oplus H)$ given by*

$$\Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \varphi_1(a) & \varphi(b) \\ \varphi(c^*)^* & \varphi_2(d) \end{pmatrix}$$

is completely positive.

Applying this technique to our completely bounded map φ and then Stinespring's theorem to the resulting operator Φ gives the required representation.

The following decomposition theorem may be obtained either as a corollary of the representation theorem or may be proved independently and then used to prove the representation theorem. It was first proved by Wittstock, and was independently proved by Haagerup.

Theorem 1.2.5 *Let φ be a linear operator from a C^*-algebra into $B(H)$. Then φ is completely bounded if and only if it is a linear combination of four completely positive operators.*

The following result will be extremely useful to us in later chapters.

Proposition 1.2.6 ([Pa] Proposition 3.7) *If X is a subspace of a C^*-algebra, and $f : X \to \mathbb{C}$ is a bounded linear functional, then f is completely bounded and $\|f\|_{cb} = \|f\|$.*

The proof is a straightforward calculation using appropriate matrices.

Finally, we introduce completely bounded bilinear operators, see [CS1], Definition 1.1.

Definition 1.2.7 *Let A and B be C^*-algebras. If φ is a bilinear operator from $A \times A$ into B, define the bilinear operator φ_n from $M_n(A) \times M_n(A)$ into $M_n(B)$ by*

$$\varphi_n((x_{ij}),(y_{ij})) = \left(\sum_{k=1}^{n} \varphi(x_{ik}, y_{kj})\right), \qquad (x_{ij}),\ (y_{ij}) \in M_n(A).$$

The bilinear operator is said to be completely bounded with completely bounded norm $\|\varphi\|_{cb}$ if

$$\|\varphi\|_{cb} = sup\{\|\varphi_n\| : n \in \mathbb{N}\}$$

is finite.

Then we have a representation theorem for completely bounded bilinear operators, proved by Christensen and Sinclair [CS1] and Paulsen and Smith [PS].

Theorem 1.2.8 *Let A be a C^*-algebra, and let φ be a bilinear operator from $A \times A$ into $B(H)$, where H is a Hilbert space. The operator φ is completely bounded if and only if there are *-representations π_1, π_2 of A on Hilbert spaces H_1, H_2 and continuous linear operators V_1, V_2, V_3 with $V_3 : H \to H_2,\ V_2 : H_2 \to H_1,\ V_1 : H_1 \to H$ such that*

$$\varphi(x,y) = V_1\pi_1(x)V_2\pi_2(y)V_3, \qquad x, y \in A,$$

and that $\|\varphi\|_{cb} = \|V_1\| \cdot \|V_2\| \cdot \|V_3\|$.

1.3 Compact operators on a Hilbert space

Here we draw together some basic facts and results about compact operators which will we need throughout the later chapters. For more details about compact operators and the other classes of operators defined here, see the books by Ringrose [Ri], Taylor [Ta], and Rickart [R].

Definition 1.3.1 *(i) If X is a Banach space and x is an element of $B(X)$ then x is said to be finite rank if the subspace $M = \{x\xi : \xi \in X\}$ of X is finite-dimensional; the dimension of M is called the rank of x. We denote by $F(X)$ the set of finite rank operators on X.*

(ii) If the norm closure of $\{x\xi : \xi \in X, \|\xi\| \leqslant 1\}$ is compact in the norm topology on X then x is said to be a compact operator on X. We denote by $K(X)$ the set of compact operators on X.

(iii) If the norm closure of $\{x\xi : \xi \in X, \|\xi\| \leqslant 1\}$ is compact in the weak topology on X then x is said to be a weakly compact operator on X.

(iv) If H is a Hilbert space and $x \in B(H)$ with polar decomposition $x = vh$, then if $f_p(t) = t^p$ by the functional calculus we may define $h^p = f_p(h)$. If $\{\xi_n\}$ is an orthonormal sequence in H then define

$$\|x\|_p = \left[\sum_n < h^p\xi_n, \xi_n >\right]^{\frac{1}{p}},$$

and this quantity is independent of the choice of basis (for details see [Ri], Chapter 2.1). Then define $C_p = \{x \in B(H) : \|x\|_p < \infty\}$. The operators in C_1 are often called the trace class operators on H and we will write $C_1 = T(H)$, and the operators in C_2 are often called the Hilbert-Schmidt operators on H and we will write $C_2 = HS(H)$.

With the above definitions it is easy to see that if H is a Hilbert space, then all the classes so defined form ideals of $B(H)$, since bounded operators map compact sets to compact sets, and $\|axb\|_p \leqslant \|a\| \cdot \|x\|_p\|b\|$. In particular, $K(H)$ is a norm closed ideal of $B(H)$. Also, $C_p \subseteq C_q$ if $1 \leqslant p \leqslant q < \infty$, since $l^p \subseteq l^q$ if $1 \leqslant p \leqslant q < \infty$. If ξ and η are vectors in a Hilbert space we denote by $\xi \otimes \eta$ the compact operator on H given by

$$\xi \otimes \eta(\lambda) = < \lambda, \eta > \xi, \qquad \lambda \in H.$$

We can define a trace Tr on the trace class operators by

$$Tr(x) = \sum_{j=1}^{\infty} < x\xi_j, \xi_j >, \qquad x \in T(H),$$

where $\{\xi_j\}$ is an orthonormal basis of H. Again, the the definition is independent of the choice of basis (see [Ri]).

We now state some well-known results about compact operators on Hilbert space.

Theorem 1.3.2 ([Ri], Theorem 1.8.7) *Suppose that H is a Hilbert space and $x \in B(H)$. Then the following are equivalent:*

(i) x is compact,

(ii) given any orthonormal basis $\{\xi_j : j \in J\}$ of H, $< x\xi_j, \xi_j > \to 0$ as $j \to \infty$,

(iii) there is a sequence (y_n) of finite rank operators on H such that $\|x - y_n\| \to 0$ as $n \to \infty$.

It should be emphasised, as we have seen in connection with the approximation property, that this theorem does not hold for all Banach spaces X.

Theorem 1.3.3 ([Ri], Theorem 1.9.3) *Suppose that (μ_n) is a decreasing sequence of positive real numbers which is either finite or infinite and convergent to 0, and that (η_n) and (ν_n) are orthonormal sequences in a Hilbert space H. Then the equation*

$$x\xi = \sum \mu_n < \xi, \eta_n > \nu_n, \qquad \xi \in H,$$

defines a compact linear operator x on H; furthermore, x has finite rank k if and only if the sequence (μ_n) terminates after just k terms.

Conversely, if x is a compact operator acting on H, then x may be expressed in the above form; the sequence (μ_n) is uniquely determined, and consists of the eigenvalues of $(x^*x)^{\frac{1}{2}}$, arranged in order of decreasing magnitude and counted according to their multiplicities.

This is proved by working with the polar decomposition of the operator.

Theorem 1.3.4 ([Ri]) *Suppose that $1 \leqslant p < \infty$, and that x is a compact operator on a Hilbert space H. Let (μ_n) be the eigenvalues of $(x^*x)^{\frac{1}{2}}$, counted according to their multiplicities. Then $x \in C_p$ if and only if $\sum |\mu_n|^p < \infty$, $\|x\|_p = (\sum |\mu_n|^p)^{\frac{1}{p}}$, and $\|\cdot\|_p$ is a norm on C_p.*

This follows from the definition and the previous theorem.

Corollary 1.3.5 *If $x \in C_p(H)$ and (μ_n) are the eigenvalues of x^*x, counted according to their multiplicities, there exist orthonormal sequences (η_n) and (ν_n) in H such that*

$$x\xi = \sum \mu_n < \xi, \eta_n > \nu_n, \qquad \xi \in H,$$

and

$$\|x\|_p = \left(\sum |\mu_n|^p\right)^{\frac{1}{p}}.$$

This is a straightforward combination of the previous two results.

We will also need the result that if H is a separable Hilbert space then $K(H)$ is the only norm-closed ideal of $B(H)$, which we prove here as we will use parts of the proof later. The proof given here follows that in [Ha], Problem 176.

Theorem 1.3.6 *If H is a separable Hilbert space, then $K(H)$ is the only non-zero proper norm-closed ideal of $B(H)$.*

Proof

Suppose that I is a non-zero closed ideal of $B(H)$. The first step is to show that I contains all the rank one operators. Note that if ξ and η are non-zero vectors, then the operator $\xi \otimes \eta$ has rank one, and every rank one operator has this form. Suppose x_0 is an operator in I, and let ξ_0 and η_0 be non-zero vectors such that $x_0\xi_0 = \eta_0$. Let y be an arbitrary operator such that $y\eta_0 = \eta$. Then

$$yx_0(\xi \otimes \xi_0)\nu = yx_0 < \nu, \xi > \xi_0 =< \nu, \xi > y\eta_0 =< \nu, \xi > \eta = (\xi \otimes \eta)\nu, \ \nu \in H,$$

that is $yx_0(\xi \otimes \xi_0) = \xi \otimes \eta$. Since I is an ideal it follows that $\xi \otimes \eta \in I$.

Thus I contains all rank one operators, and so it contains all finite rank operators, and since it is closed it contains all compact operators. Note that the separability of H is not needed for this part of the proof.

Suppose now that I contains an operator x that is not compact. Let up be the polar decomposition of x, then $p = u^*x$, so $p \in I$, and p is not compact since $x = up$. Now p is self-adjoint, so there exists an infinite-dimensional subspace M of H, invariant under p, on which p is bounded below, by ε say. Let v be an isometry from H to M. Since $pM = M$, it follows that $v^*pvM = v^*M = H$. Moreover, $v\xi \in M$ for all $\xi \in H$, so it follows that

$$\|v^*pv\xi\| = \|pv\xi\| \geqslant \varepsilon\|v\xi\| = \varepsilon\|\xi\|.$$

Now if x is a bounded linear operator that maps H one-to-one onto itself, then x is invertible (see [Ha] Problem 52, for example). We have seen that $v^*pvH = H$ so v^*pv is onto and $\|v^*pv\xi\| \geqslant \varepsilon\|\xi\|$, so v^*pv is one-to-one. Hence v^*pv is invertible. Since $v^*pv \in I$, I must contain all of $B(H)$. □

We will also need weakly compact operators, which are discussed in Dunford & Schwartz VI.4.6.

Theorem 1.3.7 *In the norm topology of $B(X)$, the weakly compact operators form a closed two-sided ideal.*

It is quite easy to show that if X is reflexive then the identity map I on X is weakly compact, and then by the above theorem, all the operators in $B(X)$ are weakly compact. In particular, if H is a Hilbert space, then the set of weakly compact operators on H is $B(H)$.

Finally, we will need a result from Rickart, about representations of $K(H)$, but first we need some definitions.

Definition 1.3.8 *If X is a linear space and A is a subalgebra of $B(X)$, then a subspace M of X is said to be invariant with respect to A if $xM \subseteq M$ for all $x \in A$.*

A representation π of a Banach algebra A on a linear space X is said to be algebraically irreducible if $\{0\}$ and X are the only subspaces which are invariant with respect to $\pi(A)$; π is said to be topologically irreducible if $\{0\}$ and X are the only closed invariant subspaces.

Theorem 1.3.9 ([R], Theorem 4.9.10) *If π is a *-representation of a C^*-algebra A on a Hilbert space H then π is topologically irreducible if and only if it is algebraically irreducible.*

The proof considers A represented on a Hilbert space and then uses the von Neumann double commutant theorem and the Kaplansky density theorem.

Hence, for representations of C*-algebras on Hilbert spaces, no confusion will arise if we omit the adverbs algebraically and topologically. Recall that the commutant A' of a subalgebra A of an algebra B is defined to be

$$A' = \{y \in B : xy = yx \ \forall x \in A\}.$$

Proposition 1.3.10 ([T], Proposition I.9.20) *If π is a representation of a C^*-algebra A on a Hilbert space H, then π is irreducible if and only if*

$$\pi(A)' = \{\lambda 1 : \lambda \in \mathbb{C}\}.$$

The proof entails calculating what commutes with the spectral projections of operators in $\pi(A)$.

Theorem 1.3.11 (D2, Corollary 4.1.5) *If π is an irreducible representation of $K(H)$ then π is unitarily equivalent to the identity representation.*

1.4 Conditional expectations

In suitable von Neumann algebras there exist very useful projections called conditional expectations. Since we will largely be working with $B(H)$ we will be able to put these to good use. The definition is from Sakai §2.6 [Sa], and is due to Umegaki [U].

Definition 1.4.1 *A conditional expectation from a C^*-algebra A onto a C^*-subalgebra B is a linear map $\mathbb{E}$ satisfying*

(i) $\mathbb{E}(1) = 1$,

(ii) $\|\mathbb{E}(x)\| \leqslant \|x\|, \qquad x \in A$,

(iii) $\mathbb{E}(x) \geqslant 0$, *if* $x \geqslant 0$,

(iv) $\mathbb{E}(axb) = a\mathbb{E}(x)b, \qquad x \in A,\ a, b \in B$,

(v) $\mathbb{E}(x)^*\mathbb{E}(x) \leqslant \mathbb{E}(x^*x), \qquad x \in A$.

Suppose now that N is a matrix subalgebra of $B(H)$. Let tr be the normalised trace on N. Then the Hahn-Banach theorem gives a state f_0 on $B(H)$ with $f_0|_N = tr$. We can then average f_0 over the unitary group of N to get a new state f;

$$f(x) = \int_{U(N)} f_0(u^*xu)\,d\mu(u), \qquad x \in B(H),$$

where μ is normalised Haar measure on $U(N)$, the unitary group of N. Then f is an N hypertrace on $B(H)$, that is, $f|_N = tr$ and

$$f(xn) = f(nx), \qquad n \in N, x \in B(H).$$

Now let $\{e_{ij}\}$ be the matrix units on N and define $\mathbb{E} : B(H) \to N$ by

$$\mathbb{E}(x) = n \sum_{i,j=1}^{n} f(e_{ij}x)e_{ji}.$$

Then $\mathbb{E}$ is a conditional expectation onto N.

2 Tensor Products of Completely Bounded Operators

In this chapter, our aim is to introduce the Haagerup and weak*-Haagerup tensor products and study some of their basic properties. Few of the results in this chapter are original: full attributions are given in the text. Generally, section 1 follows Smith's paper [S] closely and section 2 Blecher and Smith [BS], in addition culling results from the papers of Blecher, Paulsen, Effros and Ruan where necessary. Proofs are included as most of these papers are yet to be published. The proofs given usually follow those given in the original papers very closely, and I apologise for any errors introduced. The main departure from these papers is that we do not consider the results in the setting of operator spaces. As we do not consider operator spaces in the later chapters, and there is not enough space to fully prove all the necessary theorems, we ignore the matricial structures on the spaces involved: thus we show that spaces are isometrically isomorphic rather than completely isometrically isomorphic. I apologise to the above mentioned authors for this bowdlerisation of their theorems.

In section 1, some motivation for the definition of the Haagerup tensor product is given in terms of the isomorphism with the space of completely bounded operators. For more on this isomorphism see the papers of Chatterjee and Sinclair [ChS] and Chatterjee and Smith [ChSm]. We also characterise the Haagerup tensor product in terms of slice maps.

In section 2, we define the weak*-Haagerup tensor product as the dual of the Haagerup tensor product. We show that this is isometrically isomorphic to a certain space of completely bounded operators. We then emphasise the difference between the two tensor products, namely the norm convergence of the sums in the Haagerup tensor and the strong convergence of the sums in the weak*-Haagerup tensor. Finally, in the only original work in this chapter, we extend the definition of the weak*-Haagerup tensor product to C*-algebras and show that if A and B are C*-algebras then $A \otimes_{w^*h} B$ is a Banach algebra.

2.1 The Haagerup Tensor Product

The motivation behind the definition of the Haagerup tensor product is most easily seen by an example involving $m \times m$ matrices. Let $M_m(\mathbb{C})$ denote the algebra of $m \times m$ complex matrices acting on m-dimensional complex Hilbert space $\mathbb{C}^m$.

Let $\{e_{ij}\}$ be a system of matrix units for $M_m(\mathbb{C})$. Then each matrix unit may be linearly mapped to any of the m^2 matrix units. Now since the matrix units span $M_m(\mathbb{C})$, any linear operator on $M_m(\mathbb{C})$ is defined by its action on the matrix units, so this gives a total of m^4 linearly independent linear operators on $M_m(\mathbb{C})$. If $L_a : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ denotes left multiplication by $a \in M_m(\mathbb{C})$ and $R_b : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ denotes right multiplication by $b \in M_m(\mathbb{C})$, so $L_a(x) = ax$, $R_b(x) = xb$, $x \in M_m(\mathbb{C})$, then the linear span of the set $\{L_{e_{ij}} R_{e_{kl}} : 1 \leqslant i,j,k,l \leqslant m\}$ has dimension m^4, so there is an isomorphism between the linear span of this set and the set of all linear operators on $M_m(\mathbb{C})$. That is, all linear operators on $M_m(\mathbb{C})$ are sums of at most m^4 left and right multiplications.

Now let $u_1, \ldots, u_k, w_1, \ldots, w_k$ be in $M_m(\mathbb{C})$ and let φ be the linear operator on $M_m(\mathbb{C})$ given by $\varphi(x) = \sum u_j x w_j$ for $x \in M_m(\mathbb{C})$. It is easy to see that

$$\|\varphi\| \leqslant \sum \|u_j\| \cdot \|w_j\|,$$

but the Cauchy-Schwarz inequality gives the stronger inequality

$$\|\varphi\| \leqslant \left\| \sum u_j u_j^* \right\|^{\frac{1}{2}} \cdot \left\| \sum w_j^* w_j \right\|^{\frac{1}{2}}.$$

To see this let ξ, η be in $\mathbb{C}^m$; then

$$\begin{aligned} | < \varphi(x)\xi, \eta > | &= \left| \sum < x w_j \xi, u_j^* \eta > \right| \\ &\leqslant \left(\sum \|x w_j \xi\|^2 \right)^{\frac{1}{2}} \left(\sum \|u_j^* \eta\|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

by Cauchy-Schwarz. Now $\|x w_j \xi\| \leqslant \|x\| \cdot \|w_j \xi\|$ and

$$\sum \|w_j \xi\|^2 = \sum < w_j^* w_j \xi, \xi > \leqslant \| \sum w_j^* w_j \| \cdot \|\xi\|^2.$$

A similar calculation with the u_j gives

$$| < \varphi(x)\xi, \eta > | \leqslant \|x\| \cdot \left\| \sum u_j u_j^* \right\|^{\frac{1}{2}} \cdot \left\| \sum w_j^* w_j \right\|^{\frac{1}{2}} \cdot \|\xi\| \cdot \|\eta\|,$$

and so

$$\|\varphi\| \leqslant \left\|\sum u_j u_j^*\right\|^{\frac{1}{2}} \cdot \left\|\sum w_j^* w_j\right\|^{\frac{1}{2}}.$$

We can apply the same inequality to $\varphi_n = \varphi \otimes \iota_n : M_m(\mathbb{C}) \otimes M_n(\mathbb{C}) \to M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$, where ι_n is the identity on $M_n(\mathbb{C})$. Note that the norm on $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ is inherited from the natural isomorphism with $M_{mn}(\mathbb{C})$, given by considering $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ as $n \times n$ matrices with $m \times m$ matrices as elements and then ignoring the brackets. Then we get $\varphi_n = \sum L_{\iota_n \otimes u_j} R_{\iota_n \otimes w_j}$, but note that

$$u \otimes \iota_n = \begin{pmatrix} u & 0 & . & 0 \\ 0 & u & . & 0 \\ . & . & . & . \\ 0 & 0 & . & u \end{pmatrix},$$

so that for all $n \in \mathbb{N}$,

$$\begin{aligned} \|u \otimes \iota_n\| &= sup\{\|u \otimes \iota_n(\xi)\| : \xi \in \mathbb{C}^{mn}, \|\xi\| \leqslant 1\} \\ &= sup\{\|u(\xi)\| : \xi \in \mathbb{C}^m, \|\xi\| \leqslant 1\} \\ &= \|u\|. \end{aligned}$$

Thus

$$\begin{aligned} \|\varphi_n\| &\leqslant \left\|\sum (u_j \otimes \iota_n)(u_j \otimes \iota_n)^*\right\|^{\frac{1}{2}} \cdot \left\|\sum (w_j \otimes \iota_n)^*(w_j \otimes \iota_n)\right\|^{\frac{1}{2}} \\ &\leqslant \left\|\sum u_j u_j^*\right\|^{\frac{1}{2}} \cdot \left\|\sum w_j^* w_j\right\|^{\frac{1}{2}}, \end{aligned}$$

for all $n \in \mathbb{N}$, and so

$$\|\varphi\|_{cb} \leqslant \left\|\sum u_j u_j^*\right\|^{\frac{1}{2}} \cdot \left\|\sum w_j^* w_j\right\|^{\frac{1}{2}}. \tag{2}$$

Now consider the transpose map on $M_m(\mathbb{C})$; let $\tau : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ be given by

$$\tau(x_{ij}) = (x_{ij})^t = (x_{ji}), \qquad (x_{ij}) \in M_m(\mathbb{C}).$$

Then $\|\tau\| \leqslant 1$, but $\tau(\iota_m) = \iota_m$, so $\|\tau\| = 1$.

However, we have the following result of Tomiyama [To].

Theorem 2.1.1 *If τ is the transpose map on a $M_m(\mathbb{C})$, then*

$$\|\tau \otimes \iota_n\| = \begin{cases} n, & if\ n \leqslant m, \\ m, & if\ n > m. \end{cases}$$

The proof uses the fact that if A is a C*-algebra, and $(a_{ij}) \in M_n(A)$, then

$$\|(a_{ij})\| \leqslant \left(\sum_{i,j=1}^{n} \|a_{ij}\|^2 \right)^{\frac{1}{2}},$$

and then, for the case $n \leqslant m$, considers the transpose map on the matrix

$$x = \begin{pmatrix} e_{11} & e_{21} & . & e_{n1} & 0 \\ e_{12} & e_{22} & . & e_{n2} & 0 \\ . & . & . & . & . \\ e_{1n} & e_{2n} & . & e_{nn} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

in $M_m(M_n(\mathbb{C})) \cong M_n(M_m(\mathbb{C}))$, where $\{e_{ij}\}$ are the matrix units of $M_n(\mathbb{C})$.

But if we consider the action of τ on the matrix units we can show that

$$\tau = \sum_{i,j=1}^{m} L_{e_{ij}} R_{e_{ij}}.$$

Then (2) becomes

$$n = \|\tau\|_{cb} \leqslant \left\| \sum_{i,j=1}^{m} e_{ij} e_{ij}^* \right\|^{\frac{1}{2}} \cdot \left\| \sum_{i,j=1}^{m} e_{ij}^* e_{ij} \right\|^{\frac{1}{2}} = n,$$

so (2) is best possible.

Theorem 2.1.2 *If $\varphi : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ is a linear operator, then there exist $u_1, \ldots, u_k, w_1, \ldots, w_k$ in $M_m(\mathbb{C})$ with $k \leqslant 4m^2$ such that $\varphi(x) = \sum_1^k u_j x w_j$ for $x \in M_m(\mathbb{C})$ and*

$$\|\varphi\|_{cb} = \left\| \sum_1^k u_j u_j^* \right\|^{\frac{1}{2}} \cdot \left\| \sum_1^k w_j^* w_j \right\|^{\frac{1}{2}}. \tag{3}$$

This result follows from the following theorem of Haagerup and the Wittstock decomposition theorem (Theorem 1.2.5).

Theorem 2.1.3 ([H2] Theorem 2.1) *Let N be a von Neumann algebra and let F be a finite-dimensional subfactor. Let φ be a completely positive map from F to N. Then there exist $d = dim(F)$ operators $a_1, \ldots, a_d$ in N such that*

$$\varphi(x) = \sum_{i=1}^{d} a_i^* x a_i, \qquad x \in F.$$

This shows that any completely positive map on $M_m(\mathbb{C})$ may be written as a sum of m^2 left and right multiplications and then Wittstock's decomposition theorem, which says that any completely bounded map may be written as a sum of four completely positives, gives Theorem 2.1.2. The norm that occurs on the right hand side of equation (3) is the Haagerup tensor norm $\|\cdot\|_h$ on the tensor product $M_m(\mathbb{C}) \otimes M_m(\mathbb{C})$ of the C*-algebra $M_m(\mathbb{C})$ with itself. Thus the theorem shows that the natural embedding from $M_m(\mathbb{C}) \otimes M_m(\mathbb{C})$ into $CB(M_m(\mathbb{C}))$ given by $x \otimes y \mapsto L_x R_y$ is an isometry, provided that the norm on the tensor is the Haagerup norm.

The above discussion (excluding the isometry) can be generalised to any C*-algebra A by representing A faithfully on a Hilbert space H and following the same calculation as above with $\varphi(x) = \sum_1^k u_j x v_j$ for $x \in A$ where $u_1, \ldots, u_k, w_1, \ldots, w_k \in A$ and choosing $\xi, \eta \in H$. Then

$$\|\varphi\|_{cb} \leqslant \left\| \sum_1^k u_j u_j^* \right\|^{\frac{1}{2}} \cdot \left\| \sum_1^k w_j^* w_j \right\|^{\frac{1}{2}}.$$

Now let U be the $1 \times k$ matrix with $u_1, \ldots, u_k$ as elements and W be the $k \times 1$ matrix with $w_1, \ldots, w_k$ as elements. Then

$$\|U\|^2 = \|UU^*\| = \|\sum u_j u_j^*\| \text{ and } \|W\|^2 = \|W^*W\| = \|\sum w_j^* w_j\|.$$

Now φ can be written $\varphi(x) = U(x \otimes \iota_k)W$: note that $x \mapsto x \otimes \iota_k$ is just a representation of A, so by the representation theorem φ is completely bounded with $\|\varphi\|_{cb} \leqslant \|U\| \cdot \|W\|$.

In general, of course, we cannot write a completely bounded operator as such a sum since we cannot replace the representation π of A on a Hilbert space H by a simple amplification $x \mapsto x \otimes \iota_k$, but in the case where $A = K(H)$ all irreducible representations of A are unitarily equivalent to the identity representation (Theorem 1.3.11). This leads to the following theorem which was first proved in an unpublished manuscript of Haagerup [H4]. Haagerup was concerned with module maps and his work was extended by Effros [E2], Effros and Kishimoto [EK] and Smith [S]. We are not concerned with module maps so we do not state the theorem in that setting, but our approach is fundamentally the same as Smith's.

[EK 1.3, CS1 5.7]

Theorem 2.1.4 *If $\varphi : K(H) \to B(H)$ is a completely bounded operator then there exist sequences $\{a_j\}$ and $\{b_j\}$ in $B(H)$ such that $\sum_j a_j a_j^*$, $\sum_j b_j^* b_j$ converge strongly in*

$B(H)$,

$$\left\|\sum_j a_j a_j^*\right\|^{\frac{1}{2}} \cdot \left\|\sum_j b_j^* b_j\right\|^{\frac{1}{2}} = \|\varphi\|_{cb}^2,$$

and

$$\varphi(x) = \sum_j a_j x b_j, \qquad x \in K(H).$$

If φ is a completely bounded operator from $K(H)$ to $B(H)$ then, since $K(H)^{**} = B(H)$ [Pe], φ^{**} is a weak*-continuous completely bounded operator from $B(H)$ to $B(H)^{**}$. Composing φ^{**} with the weak*-continuous projection from $B(H)^{**}$ to $B(H)$ gives a weak*-continuous completely bounded operator from $B(H)$ to $B(H)$. Conversely any weak*-continuous completely bounded operator on $B(H)$ defines a completely bounded operator on $K(H)$ by restriction. Thus the above theorem may be considered as a theorem about completely bounded weak*-continuous operators on $B(H)$, as it was originally by Haagerup.

This theorem suggests that a similar isometry as we found from $M_m(\mathbb{C}) \otimes M_m(\mathbb{C})$ to $CB(M_m(\mathbb{C}))$, might exist from $K(H) \otimes K(H)$ to $CB(K(H))$, where the norm on the tensor is the Haagerup norm, defined below as an analogue of the $M_m(\mathbb{C})$ case; this is a prime motivation for studying the Haagerup norm.

Definition 2.1.5 *If A and B are C^*-algebras, the Haagerup norm $\|\cdot\|_h$ on the algebraic tensor product $A \odot B$ of A and B is defined by*

$$\|u\|_h = inf\left\{ \left\|\sum_{j=1}^n a_j a_j^*\right\|^{\frac{1}{2}} \cdot \left\|\sum_{j=1}^n b_j^* b_j\right\|^{\frac{1}{2}} : u = \sum_{j=1}^n a_j \otimes b_j, a_j \in A, b_j \in B, n \in \mathbb{N} \right\}$$

for all $u \in A \odot B$. The completion of $A \odot B$ in $\|\cdot\|_h$ is called the Haagerup tensor product $A \otimes_h B$ of A and B.

We can also define the Haagerup tensor product of two subspaces X and Y of C*-algebras A and B respectively, to be the completion of $X \odot Y$ in the Haagerup norm inherited from $A \otimes_h B$.

There are four possible multiplications on $A \otimes_h B$:

$$(a \otimes b)(c \otimes d) \quad = \quad ac \otimes bd,$$

$$
\begin{aligned}
(a \otimes b)(c \otimes d) &= ac \otimes db, \\
(a \otimes b)(c \otimes d) &= ca \otimes bd, \\
(a \otimes b)(c \otimes d) &= ca \otimes db,
\end{aligned}
$$

and $A \otimes_h B$ is an algebra under all of them. We shall use the second as it emphasises the relation with operators.

Blecher [B1] includes the following calculation due to R. R. Smith that the Haagerup norm is an algebra norm, that is that if $u, v \in A \otimes_h B$ then $\|u \cdot v\|_h \leqslant \|u\|_h \cdot \|v\|_h$. If T is a self-adjoint operator on a Hilbert space H and $S_1, \ldots, S_n$ are bounded operators on H, then

$$
\begin{aligned}
\left\|\sum_{j=1}^{n} S_j T S_j^*\right\| &= sup \left\{ \left| \sum_{j=1}^{n} < TS_j^*(\xi), S_j^*(\xi) > \right| : \xi \in H, \|\xi\| \leqslant 1 \right\} \\
&\leqslant \|T\| \cdot sup \left\{ \sum_{j=1}^{n} | < S_j^*(\xi), S_j^*(\xi) > | : \xi \in H, \|\xi\| \leqslant 1 \right\} \\
&= \|T\| \cdot \left\| \sum_{j=1}^{n} S_j S_j^* \right\|.
\end{aligned}
$$

Then if $u = \sum_{i=1}^{n} a_i \otimes b_i$ and $v = \sum_{j=1}^{m} x_j \otimes y_j$ are in the algebraic tensor product of two C*-algebras, then

$$
\begin{aligned}
\|u \cdot v\|_h &\leqslant \left\| \sum_{i=1}^{n} \sum_{j=1}^{m} a_i x_j x_j^* a_i^* \right\|^{\frac{1}{2}} \cdot \left\| \sum_{i=1}^{n} \sum_{j=1}^{m} y_j^* b_i^* b_i y_j \right\|^{\frac{1}{2}} \\
&\leqslant \left\| \sum_{i=1}^{n} a_i a_i^* \right\|^{\frac{1}{2}} \cdot \left\| \sum_{i=1}^{n} b_i^* b_i \right\|^{\frac{1}{2}} \cdot \left\| \sum_{j=1}^{m} x_j x_j^* \right\|^{\frac{1}{2}} \cdot \left\| \sum_{j=1}^{m} y_j^* y_j \right\|^{\frac{1}{2}},
\end{aligned}
$$

and so $\|u.v\|_h \leqslant \|u\|_h \|v\|_h$.

We will also need the following definition, which is again due to Smith [S], but first we must define some notation. For a Hilbert space H,we denote by H^∞ the Hilbert space $\oplus_{j=1}^{\infty} H$, and we denote the identity map on this space by ι_∞. If $\underline{\lambda} \in l^2$, $\underline{\lambda} = (\lambda_j)$ say, and if $\underline{b} = (b_1, b_2, \ldots)$ is a sequence of elements of $B(H)$ where $\sum_j b_j^* b_j$ converges strongly in $B(H)$, then we will denote by $\underline{\lambda}.\underline{b}$ the sum $\sum_j \lambda_j b_j$ which converges in norm in $B(H)$, as can be seen from the following lemma.

Lemma 2.1.6 *If $\underline{\lambda} \in l^2$ and $\{b_j\}$ is a sequence in $B(H)$ with $\sum_j b_j^* b_j$ strongly convergent then $\sum_j \lambda_j b_j$ is norm convergent.*

Proof

Suppose $\xi \in H$ with $\|\xi\| \leqslant 1$. Then for $m, n \in \mathbb{N}$,

$$\begin{aligned}
\left\|\sum_{j=m}^{n} \lambda_j b_j \xi\right\| &\leqslant \sum_{j=m}^{n} |\lambda_j| \cdot \|b_j \xi\| \\
&\leqslant \left(\sum_{j=m}^{n} |\lambda_j|^2\right)^{\frac{1}{2}} \left(\sum_{j=m}^{n} < b_j\xi, b_j\xi >\right)^{\frac{1}{2}} \\
&\leqslant \left(\sum_{j=m}^{n} |\lambda_j|^2\right)^{\frac{1}{2}} \left(\sum_{j=m}^{n} < b_j^* b_j\xi, \xi >\right)^{\frac{1}{2}} \\
&\leqslant \left(\sum_{j=m}^{n} |\lambda_j|^2\right)^{\frac{1}{2}} \left\|\sum_{j=m}^{n} b_j^* b_j\right\|^{\frac{1}{2}} \\
&\leqslant \left(\sum_{j=m}^{n} |\lambda_j|^2\right)^{\frac{1}{2}} \left\|\sum_{j=1}^{\infty} b_j^* b_j\right\|^{\frac{1}{2}},
\end{aligned}$$

but $\{(\sum_{j=1}^{n} |\lambda_j|^2)^{\frac{1}{2}}\}$ is a Cauchy sequence so $\{\sum_{j=1}^{n} \lambda_j b_j\}$ is a Cauchy sequence. □

We can identify a row of elements, $(a_1, a_2, \ldots)$, of $B(H)$ with an element $a \in B(H^\infty, H)$ provided $\sum_j a_j a_j^*$ converges strongly in $B(H)$ and a column of elements $(b_1, b_2, \ldots)^t$ of $B(H)$ with an element b of $B(H, H^\infty)$ provided $\sum b_j^* b_j$ converges strongly in $B(H)$. Then if $u \in B(l^2)$, with columns u_j and rows v_j, we can define au to be the element of $B(H^\infty, H)$ with components $(a_1, a_2, \ldots).u_j$ and bu to be the element of $B(H, H^\infty)$ with components $v_j.(b_1, b_2, \ldots)^t$. If $\{a_j\}$ is a sequence of elements of $B(H)$, let $[a_j]$ denote the closed linear span of the a_j.

Definition 2.1.7 *Let W be a norm closed subspace of $B(H)$. A set of operators $\{a_j : j \in \mathbb{N}\}$, with $a_j \in B(H), \sum_j a_j a_j^*$ strongly convergent in $B(H)$, is said to be strongly independent over W if $\underline{\lambda} = 0$ whenever $\underline{\lambda} \in l^2$ and $\underline{\lambda}.\underline{a} \in W$, where $\underline{a} = (a_1, a_2, \ldots)$. If W is the zero subspace then we will say simply that the set $\{a_j\}$ is strongly independent.*

The following lemma of Smith's will be very useful to us in later chapters.

Lemma 2.1.8 ([S] Lemma 4.1) *Let $a \in B(H^\infty, H)$ and $b \in B(H, H^\infty)$ have components $a_j, b_j \in B(H)$ respectively and let W be a closed subspace of $B(H)$. Then there exist unitaries $u_1, u_2 \in B(l^2)$ and disjoint decompositions $M_1 \cup M_2 \cup M_3, N_1 \cup N_2 \cup N_3$ of $\mathbb{N}$ such that the components $\tilde{a}_j$ and $\tilde{b}_j$ of $\tilde{a} = au_2$ and $\tilde{b} = u_1 b$ satisfy* *(i)* $\tilde{a}_j = 0$ *for* $j \in M_1$, $\tilde{b}_j = 0$ *for* $j \in N_1$,

(ii) $\tilde{a}_j \in W \cap [a_j : j \in \mathbb{N}]$ *for* $j \in M_2$, $\tilde{b}_j \in W \cap [b_j : j \in \mathbb{N}]$ *for* $j \in N_2$, *and* $\{\tilde{a}_j\}_{j \in M_2}, \{\tilde{b}_j\}_{j \in N_2}$ *are strongly independent,*

(iii) $\tilde{a}_j \in [a_j : j \in \mathbb{N}]$ *for* $j \in M_3$, $\tilde{b}_j \in [b_j : j \in \mathbb{N}]$ *for* $j \in N_3$, *and* $\{\tilde{a}_j\}_{j \in M_3}, \{\tilde{b}_j\}_{j \in N_3}$ *are strongly independent over* W,

(iv) $\|\tilde{a}\| = \|a\|$ *and* $\|\tilde{b}\| = \|b\|$,

(v) if W *is finite dimensional then* M_2 *and* N_2 *are finite sets.*

Proof

The fourth part follows from the fact that u_1 and u_2 are unitaries. We consider the case of column matrices, that is for $b \in B(H, H^\infty)$; having done this we may apply it to $a^* \in B(H, H^\infty)$ and the subspace W^* to obtain the result for row matrices.

Decompose l^2 as an orthogonal sum $L_1 \oplus L_2 \oplus L_3$ where $L_1 = \{\underline{\lambda} \in l^2 : \underline{\lambda}.b = 0\}$, L_2 is the orthogonal complement of L_1 in $\{\underline{\lambda} \in l^2 : \underline{\lambda}.b \in W\}$ and L_3 is the orthogonal complement of $L_1 \oplus L_2$ in l^2. Let $\{\underline{\alpha}_j\}$ be an orthonormal basis for l^2, constructed as a union of the orthonormal bases of L_1, L_2 and L_3. Then there is a decomposition $\mathbb{N} = N_1 \cup N_2 \cup N_3$ with $\{\underline{\alpha}_j\}_{j \in N_r}$ a basis for L_r, $1 \leqslant r \leqslant 3$. Let u_1 be the unitary matrix with j^{th}row $\underline{\alpha}_j$ and let $\tilde{b} = u_1 b$. Then $\tilde{b}_j = \underline{\alpha}_j.b_j$, so $\tilde{b}_j \in [b_j]$.

By the definitions of the sets N_r, $1 \leqslant r \leqslant 3$, $\tilde{b}_j = 0$ for $j \in N_1$, and $\tilde{b}_j \in W$ for $j \in N_2$. We now show that $\{\tilde{b}_j : j \in N_3\}$ is strongly independent over W. The proof that $\{\tilde{b}_j : j \in N_2\}$ is strongly independent is similar. Suppose that $\lambda_j \in \mathbb{C}$, for $j \in N_3$, $\sum_{j \in N_3} |\lambda_j|^2 < \infty$ and $\sum_{j \in N_3} \lambda_j \tilde{b}_j \in W$. Then $\left(\sum_{j \in N_3} \lambda_j \underline{\alpha}_j\right).b \in W$ and so $\sum_{j \in N_3} \lambda_j \underline{\alpha}_j \in L_1 \oplus L_2$. But $\{\underline{\alpha}_j : j \in N_3\}$ is an orthonormal basis for L_3 and L_3 is orthogonal to $L_1 \oplus L_2$, so $\lambda_j = 0$ for $j \in N_3$. Thus properties *(i)–(iii)* follow.

Suppose now that W is finite-dimensional, say $dim\ W = d$, and suppose that the cardinality of N_2 is greater than d. Choose integers $i_1, \ldots, i_{d+1}$ in N_2. Then $\underline{\alpha}_{i_r}.b \in W$ for $1 \leqslant r \leqslant d+1$ and so there must exist $\lambda_1, \ldots, \lambda_{d+1} \in \mathbb{C}$ not all zero with

$$\sum_{r=1}^{d+1} \lambda_r \underline{\alpha}_{i_r}.b = 0.$$

But then, by definition of L_1, $\sum_{r=1}^{d+1} \lambda_r \underline{\alpha}_{i_r} \in L_1$. Now we chose i_r such that $\underline{\alpha}_{i_r} \in L_2$, so this contradicts the disjointness of L_1 and L_2. Hence N_2 has at most d elements and so (v) follows. $\square$

Corollary 2.1.9 ([S] Corollary 4.2) *If $a \in B(H^\infty, H)$, $b \in B(H, H^\infty)$, $c \in B(H^n, H)$ and $d \in B(H, H^n)$ satisfy*

$$\sum_{j=1}^{\infty} a_j x b_j - \sum_{j=1}^{n} c_j x d_j = 0, \qquad x \in K(H),$$

and $\|a\|$, $\|b\| \leqslant 1$, then there exist m *$\tilde{a} \in B(H^m, H)$ and $\tilde{b} \in B(H, H^m)$ such that*

$$\sum_{j=1}^{m} \tilde{a}_j x \tilde{b}_j - \sum_{j=1}^{n} c_j x d_j = 0, \qquad x \in K(H),$$

$\|\tilde{a}\|$, $\|\tilde{b}\| \leqslant 1$ and $\tilde{a}_j \in [c_j] \cap [a_j]$, $\tilde{b}_j \in [d_j] \cap [b_j]$.

Proof

Apply Lemma 2.1.8 with $W = [d_j]$. Then there exists a unitary matrix u and sets of integers N_1, N_2 and N_3 such that the components of $b' = ub$ satisfy

(i) $b'_j = 0$, $j \in N_1$,

(ii) $b'_j \in [d_j] \cap [b_j]$, $j \in N_2$, a finite set of m elements,

(iii) $\{b'_j\}_{j \in N_2}$ are strongly independent and $\{b'_j\}_{j \in N_3}$ are strongly independent over $[d_j]$.

Put $a' = au^*$ and notice that

$$a(x \otimes \iota_\infty)b = a(x \otimes \iota_\infty)u^*ub = au^*(x \otimes \iota_\infty)ub = a'(x \otimes \iota_\infty)b',$$

and then it follows from the hypothesis that

$$\sum_{j=1}^{\infty} a'_j x b'_j - \sum_{j=1}^{n} c_j x d_j = 0.$$

But by (i), $b'_j = 0$ for $j \in N_1$, so we have

$$\sum_{j \in N_2} a'_j x b'_j + \sum_{j \in N_3} a'_j x b'_j - \sum_{j=1}^{n} c_j x d_j = 0. \tag{4}$$

Now let $\xi_1, \ldots, \xi_4$ be any elements of H, and put $x = \xi_1 \otimes \xi_2$ in (4) and take an inner product with ξ_3 and ξ_4 to obtain

$$\sum_{j \in N_2 \cup N_3} < b'_j\xi_3, \xi_2 >< a'_j\xi_1, \xi_4 > - \sum_{j=1}^{n} < d_j\xi_3, \xi_2 >< c_j\xi_1, \xi_4 >= 0.$$

Since ξ_3 and ξ_2 can vary, we must have

$$\sum_{j \in N_3} < a'_j\xi_1, \xi_4 > b'_j = \sum_{j=1}^{n} < c_j\xi_1, \xi_4 > d_j - \sum_{j \in N_2} < a'_j\xi_1, \xi_4 > b'_j \in [d_j],$$

by the choice of N_2. But $\{b'_j : j \in N_3\}$ is strongly independent over $[d_j]$ so $< a'_j\xi_1, \xi_4 >= 0$ for $j \in N_3$. Again, ξ_1 and ξ_4 were arbitrary so $a'_j = 0$ for $j \in N_3$. Then (4) becomes

$$\sum_{j \in N_2} a'_j x b'_j - \sum_{j=1}^{n} c_j x d_j = 0.$$

Let $\{j_1, \ldots, j_m\}$ be the elements of N_2 and define $\tilde{a}_k = a'_{j_k}$ and $\tilde{b}_k = b'_{j_k}$. Then $\tilde{a} \in B(H^m, H)$, $\tilde{b} \in B(H, H^m)$ and

$$\sum_{j=1}^{m} \tilde{a}_j x \tilde{b}_j - \sum_{j=1}^{n} c_j x d_j = 0. \tag{5}$$

Now $\|\tilde{a}\| \leqslant \|au^*\| = \|a\| \leqslant 1$ and similarly $\|\tilde{b}\| \leqslant 1$. Finally we must show that $\tilde{a}_j \in [c_j]$ as $\tilde{a}_j \in [a_j]$ by definition. The set $\{\tilde{b}_j\}$ is linearly independent and is contained in $[d_j]$ by (iii), and so extends to a basis $\{\tilde{b}_j\}_{j=1}^{r}$ for $[d_j]$. Then, for some λ_{jk}, $1 \leqslant j, k \leqslant r$, $d_j = \sum_{k=1}^{r} \lambda_{jk}\tilde{b}_k$ and hence if $\tilde{c_k} = \sum_{j=1}^{n} \lambda_{jk} c_j$, (5) becomes

$$\sum_{j=1}^{m} \tilde{a}_j x \tilde{b}_j - \sum_{k=1}^{r} \tilde{c}_k x \tilde{b}_k = 0,$$

where $\tilde{c}_j \in [c_j]$. Thus

$$\sum_{j=1}^{m} (\tilde{a}_j - \tilde{c}_j) x \tilde{b}_j - \sum_{j=m+1}^{r} \tilde{c}_j x \tilde{b}_j = 0,$$

but, choosing $\xi_1, \ldots, \xi_4 \in H$ as above and taking an inner product, we get

$$\sum_{j=1}^{m} < (\tilde{a}_j - \tilde{c}_j)\xi_1, \xi_4 >< \tilde{b}_j\xi_3, \xi_2 > - \sum_{j=m+1}^{r} < \tilde{c}_j\xi_1, \xi_4 >< \tilde{b}_j\xi_3, \xi_2 >= 0,$$

and, since ξ_2 and ξ_3 were arbitrary,

$$\sum_{j=1}^{m} < (\tilde{a}_j - \tilde{c}_j)\xi_1, \xi_4 > \tilde{b}_j - \sum_{j=m+1}^{r} < \tilde{c}_j\xi_1, \xi_4 > \tilde{b}_j = 0,$$

giving a linear dependence of the $\tilde{b}_j$. But these were linearly independent so we conclude that $\tilde{a}_j - \tilde{c}_j = 0$ for $j = 1, \ldots, m$ and hence that $\tilde{a}_j \in [c_j]$ as required. □

Now we can see that the required isometry is true.

Theorem 2.1.10 ([S] Theorem 4.3) *The map $v \mapsto \varphi_v$ from $B(H) \otimes_h B(H)$ into $CB(K(H))$ is an isometry.*

Proof

Suppose $v \in B(H) \odot B(H)$, $v = \sum_{j=1}^{n} a_j \otimes b_j$, and $\|v\|_h \leqslant 1$. Then, by definition of the Haagerup norm, given $\varepsilon > 0$, there is a representation $v = \sum_{j=1}^{n} \tilde{a}_j \otimes \tilde{b}_j$ with $\|\sum \tilde{a}_j \tilde{a}_j^*\| \leqslant 1 + \varepsilon$ and $\|\sum \tilde{b}_j^* \tilde{b}_j\| \leqslant 1 + \varepsilon$. Then

$$\begin{aligned} \|\varphi_v\|_{cb} &= \|\sum_{j=1}^{n} L_{\tilde{a}_j} R_{\tilde{b}_j}\|_{cb} \\ &\leqslant \left\|\sum_{j=1}^{n} \tilde{a}_j \tilde{a}_j^*\right\|^{\frac{1}{2}} \cdot \left\|\sum_{j=1}^{n} \tilde{b}_j^* \tilde{b}_j\right\|^{\frac{1}{2}} \\ &\leqslant 1 + \varepsilon, \end{aligned}$$

and so we only need to establish the reverse inequality.

Let $v = \sum_{j=1}^{n} c_j \otimes d_j \in B(H) \odot B(H)$ and suppose that $\|\varphi_v\|_{cb} = 1$. Then by Theorem 2.1.4, there exist $a \in B(H^\infty, H)$, $b \in B(H, H^\infty)$, each of norm one with

$$\varphi_v(x) = a(x \otimes \iota_\infty)b, \qquad x \in K(H),$$

or equivalently

$$\sum_{j=1}^{\infty} a_j x b_j - \sum_{j=1}^{n} c_j x d_j = 0.$$

By Corollary 2.1.7, there exist $\tilde{a} \in B(H^m, H)$ and $\tilde{b} \in B(H, H^m)$ satisfying $\|\tilde{a}\|, \|\tilde{b}\| \leqslant 1$ such that

$$\sum_{j=1}^{m} \tilde{a}_j x \tilde{b}_j - \sum_{j=1}^{n} \tilde{c}_j x \tilde{d}_j = 0.$$

Hence $v = \sum_{j=1}^{m} \tilde{a}_j \otimes \tilde{b}_j$ and $\|v\|_h = \|\tilde{a}\| \cdot \|\tilde{b}\| \leqslant 1$. □

Putting together Theorem 2.1.10 and Theorem 2.1.4, we see that if $v \in B(H) \otimes_h B(H)$, then the Haagerup norm of v is attained, that is there exist sequences $\{a_j\}$ and $\{b_j\}$

in $B(H)$ with

$$\|v\|_h = \left\|\sum a_j a_j^*\right\|^{\frac{1}{2}} \cdot \left\|\sum b_j^* b_j\right\|^{\frac{1}{2}}.$$

We also need an analogy of Corollary 2.1.7 for infinite sums. We identify $H^\infty \oplus H^\infty$ with H^∞ and regard H^n as a subspace of H^∞.

Lemma 2.1.11 ([S], Lemma 4.4) *Suppose that operators $a, c \in B(H^\infty, H)$, $b, d \in B(H, H^\infty), e \in B(H^n, H)$ and $f \in B(H, H^n)$ satisfy*

$$a(x \otimes \iota_\infty)b + c(x \otimes \iota_\infty)d - e(x \otimes \iota_\infty)f = 0, \qquad x \in K(H), \tag{6}$$

and $\|a\|, \|b\| \leqslant 1$, $\|c\|, \|d\| \leqslant \varepsilon < 1$. Then there exists $m \in \mathbb{N}$ and operators $\tilde{a}, \tilde{c} \in B(H^m, H)$ and $\tilde{b}, \tilde{d} \in B(H, H^m)$ with the following properties:

(i) $\tilde{a}_j \in [a_j]$ and $\tilde{b}_j \in [b_j]$,

(ii) $\|\tilde{a}\|, \|\tilde{b}\| \leqslant 1$ and $\|\tilde{c}\|, \|\tilde{d}\| \leqslant (3\varepsilon)^{\frac{1}{2}}$,

(iii) $\tilde{a}(x \otimes \iota_\infty)\tilde{b} + \tilde{c}(x \otimes \iota_\infty)\tilde{d} - e(x \otimes \iota_\infty)f = 0, \qquad x \in K(H)$.

Proof

Equation (6) may be written

$$(a \oplus c)(x \otimes \iota_\infty)(b \oplus d) - e(x \otimes \iota_\infty)f = 0.$$

By Lemma 2.1.8 and the proof of Corollary 2.1.9 with $W = [f_j]$, there exists a unitary matrix u such that $(a \oplus c)u^*$ and $u(b \oplus d)$ have only finitely many simultaneously non-zero components. Thus there exists a finite rank diagonal projection $p \in B(l^2)$ such that

$$(a \oplus c)(x \otimes \iota_\infty)(b \oplus d) = (a \oplus c)u^*(x \otimes \iota_\infty)u(b \oplus d) = (a \oplus c)u^*p(x \otimes \iota_\infty)pu(b \oplus d).$$

Write $\tilde{a} = (a \oplus 0)u^*p$, $\tilde{b} = pu(b \oplus 0)$. These matrices have only finitely many non-zero entries, the components lie in $[a_j]$ and $[b_j]$ respectively and $\|\tilde{a}\|, \|\tilde{b}\| \leqslant 1$. In addition if φ is the map given by

$$\begin{aligned}
\varphi(x) &= (a \oplus c)u^*p(x \otimes \iota_\infty)pu(b \oplus d) - \tilde{a}(x \otimes \iota_\infty)\tilde{b} \\
&= (0 \oplus c)u^*p(x \otimes \iota_\infty)pu(b \oplus 0) + (a \oplus 0)u^*p(x \otimes \iota_\infty)pu(0 \oplus d) \\
&\quad + (0 \oplus c)u^*p(x \otimes \iota_\infty)pu(0 \oplus d),
\end{aligned}$$

then

$$\begin{aligned}
\|\varphi\|_{cb} &\leqslant \|(0\oplus c)u^*p\|\cdot\|pu(b\oplus 0)\| + \|(a\oplus 0)u^*p\|\cdot\|pu(0\oplus d)\| \\
&\quad +\|(0\oplus c)u^*p\|\cdot\|pu(0\oplus d)\| \\
&\leqslant \|c\|\cdot\|\tilde{b}\| + \|\tilde{a}\|\cdot\|d\| + \|c\|\cdot\|d\| \\
&< \varepsilon+\varepsilon+\varepsilon^2 \leqslant 3\varepsilon,
\end{aligned}$$

and so may be represented as $\tilde{c}(x\otimes\iota_\infty)\tilde{d}$ where $\|\tilde{c}\|$, $\|\tilde{d}\| \leqslant (3\varepsilon)^{\frac{1}{2}}$, and $\tilde{c}$ and $\tilde{d}$ have only finitely many non-zero entries. Thus

$$\tilde{a}(x\otimes\iota_\infty)\tilde{b} + \tilde{c}(x\otimes\iota_\infty)\tilde{d} = (a\oplus c)(x\otimes\iota_\infty)(b\oplus d),$$

and making this substitution in (6) gives *(iii)*. □

We can now obtain a useful characterisation of the Haagerup tensor product. First we need to define the slice maps. If A and B are C*-algebras represented on a Hilbert space H,and $\psi\in B(H)^*$, then define $R_\psi : A\odot B\to B$ on elementary tensors by

$$R_\psi\left(\sum_j a_j\otimes b_j\right) = \sum_j \psi(a_j)b_j,$$

and $L_\psi : A\odot B\to A$ by

$$L_\psi\left(\sum_j a_j\otimes b_j\right) = \sum_j \psi(b_j)a_j.$$

Note that for $\xi\in H$ with $\|\xi\|\leqslant 1$

$$\begin{aligned}
<\sum_j \psi(a_j)b_j\xi,\xi> &\leqslant \sum_j |\psi(a_j)|\cdot\|b_j\xi\| \\
&\leqslant \left(\sum_j|\psi(a_j)|^2\right)^{\frac{1}{2}}\left(\sum_j\|b_j\xi\|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_j|\psi(a_j)|^2\right)^{\frac{1}{2}}\left(\sum_j <b_j\xi,b_j\xi>\right)^{\frac{1}{2}} \\
&= \left(\sum_j|\psi(a_j)|^2\right)^{\frac{1}{2}}\left(<\sum_j b_j^*b_j\xi,\xi>\right)^{\frac{1}{2}} \\
&\leqslant \left(\sum_j|\psi(a_j)|^2\right)^{\frac{1}{2}}\left\|\sum_j b_j^*b_j\right\|^{\frac{1}{2}},
\end{aligned}$$

and that

$$\left\|\psi_n\begin{pmatrix} a_1\dots a_n \\ 0 \end{pmatrix}\right\|^2 = \left\|\begin{pmatrix} \psi(a_1)\dots\psi(a_n) \\ 0 \end{pmatrix}\right\|^2 = \sum_j |\psi(a_j)|^2,$$

so if ψ is completely bounded then $\sum_j |\psi(a_j)|^2 < \infty$. Now by Proposition 1.2.6 all linear functionals are completely bounded with $\|\psi\|_{cb} = \|\psi\|$ so we have $\|L_\psi\| \leqslant \|\psi\|$. Hence L_ψ is bounded and so may be extended uniquely to a map on $A \otimes_h B$.

From Lemma 2.1.6, we can see that if $\psi \in B(H)^*$ then $L_\psi(u)$ is norm convergent for all $u \in A \otimes_h B$, and similarly $R_\psi(u)$ is norm convergent.

If ξ_1 and ξ_2 are vectors in H then let L_{12} and R_{12} denote respectively the left and right slice maps on $B(H) \otimes_h B(H)$ with respect to the vector functional $< \cdot\xi_1, \xi_2 >$ on $B(H)$. If $u = \sum_{j=1}^n a_j \otimes b_j \in B(H) \odot B(H)$ then for $\xi_3, \xi_4 \in H$,

$$\begin{aligned} \langle L_{12}(u)\xi_3, \xi_4\rangle &= \left\langle \sum_j a_j < b_j\xi_1, \xi_2 > \xi_3, \xi_4 \right\rangle \\ &= \left\langle \sum_j (a_j\xi_3 \otimes \xi_2) b_j \xi_1, \xi_4 \right\rangle \\ &= \langle \varphi_u(\xi_3 \otimes \xi_2)\xi_1, \xi_4\rangle, \end{aligned}$$

and by continuity this also holds for $u \in B(H) \otimes_h B(H)$. In the same way

$$\langle R_{12}(u)\xi_3, \xi_4\rangle = \langle \varphi_u(\xi_1 \otimes \xi_4)\xi_3, \xi_2\rangle. \tag{7}$$

Theorem 2.1.12 ([S] Theorem 4.5) *Let $v \in B(H) \otimes_h B(H)$, and let E and F be closed subspaces of $B(H)$. Then the following statements are equivalent.*

(i) $v \in E \otimes_h F$,

(ii) $R_\psi(v) \in F$ and $L_\psi(v) \in E$ for all $\psi \in B(H)^*$,

(iii) φ_v has a representation

$$\varphi_v(x) = e x^\infty f, \qquad x \in K(H),$$

where $e \in B(H^\infty, H)$ with components in E, $f \in B(H, H^\infty)$ with components in F, and $\|e\| = \|f\| = \|v\|_h^{\frac{1}{2}}$.

Proof

(i)⇒(ii) is clear from the definition of the slice maps, since the image of the slice maps converge in norm. Suppose now that *(ii)* is true, and assume without loss of generality that $\|v\|_h = 1$. Then $\|\varphi_v\|_{cb} = 1$ by Theorem 2.1.10, and so φ_v has a representation

$$\varphi_v(x) = e(x \otimes \iota_\infty)f, \qquad x \in K(H),$$

where $\|e\| = \|f\| = 1$. By Lemma 2.1.8, there exists a unitary matrix u and a decomposition $\mathbb{N} = N_1 \cup N_2 \cup N_3$ so that, writing $\tilde{e} = eu^*$ and $\tilde{f} = uf$,

$$\varphi_v(x) = \tilde{e}(x \otimes \iota_\infty)\tilde{f}, \qquad x \in K(H),$$

$\tilde{f}_j = 0$ for $j \in N_1$, $\tilde{f}_j \in F$ for $j \in N_2$ and $\{\tilde{f}_j\}_{j \in N_3}$ is a strongly independent set over F. Then, using (7),

$$\begin{aligned}\langle R_{12}(v)\xi_3, \xi_4\rangle &= \langle \varphi_v(\xi_1 \otimes \xi_4)\xi_3, \xi_2\rangle \\ &= \sum_{j \in N_2} \langle \tilde{e}_j\xi_1, \xi_2\rangle \left\langle \tilde{f}_j\xi_3, \xi_4\right\rangle + \sum_{j \in N_3} \langle \tilde{e}_j\xi_1, \xi_2\rangle \left\langle \tilde{f}_j\xi_3, \xi_4\right\rangle .\end{aligned}$$

Since $R_{12}(v) \in F$ by hypothesis and $\tilde{f}_j \in F$ for $j \in N_2$ by construction this implies that

$$\sum_{j \in N_3} \langle \tilde{e}_j\xi_1, \xi_2\rangle \tilde{f}_j \in F,$$

but $\{\tilde{f}_j\}_{j \in N_3}$ is strongly independent over F, so $< \tilde{e}_j\xi_1, \xi_2 >= 0$ for $j \in N_3$, and since ξ_1, ξ_2 were arbitrary, $\tilde{e}_j = 0$ for $j \in N_3$. Thus

$$\varphi_v(x) = \sum_{j \in N_2} \tilde{e}_j x \tilde{f}_j,$$

which has the form

$$\varphi_v(x) = \tilde{e}(x \otimes \iota_\infty)\tilde{f}, \qquad x \in K(H),$$

where the components of $\tilde{f}$ lie in F. The same argument can then be applied on the left, using Lemma 2.1.8 and the hypothesis that $L_{12} \in E$, and observing that if f has components in F then so does uf for any unitary matrix $u \in B(l^2)$. The norm estimates follow from the construction. Thus *(ii)* implies *(iii)*.

Now suppose that $\|v\|_h = 1$ and

$$\varphi_v(x) = e(x \otimes \iota_\infty)f, \qquad x \in K(H),$$

where the components of e and f lie in E and F respectively. Let $0 < \varepsilon < 1$ and choose $v_0 = \sum_{j=1}^{n} a_j \otimes b_j \in B(H) \odot B(H)$ with $\|v - v_0\|_h \leqslant \varepsilon^2$. Put $v_1 = v_0 - v$. Then $\|v_1\|_h \leqslant \varepsilon^2$ so $\|\varphi_{v_1}\|_{cb} \leqslant \varepsilon^2$, by Theorem 2.1.10, and thus φ_{v_1} may be represented by

$$\varphi_{v_1}(x) = c(x \otimes \iota_\infty)d, \qquad x \in K(H),$$

where $\|c\|, \|d\| \leqslant \varepsilon$. Since $v + v_1 - v_0 = 0$, we have

$$\sum_{j=1}^{\infty} e_j x f_j + \sum_{j=1}^{\infty} c_j x d_j - \sum_{j=1}^{n} a_j x b_j = 0, \qquad x \in K(H).$$

By Lemma 2.1.11 there exist $\tilde{e} \in B(H^m, H), \tilde{f} \in B(H, H^m), \tilde{c} \in B(H^\infty, H)$, $\tilde{d} \in B(H, H^\infty)$ such that $\|\tilde{e}\|, \|\tilde{f}\| \leqslant 1, \|\tilde{c}\|, \|\tilde{d}\| \leqslant (3\varepsilon)^{\frac{1}{2}}$,

$$\tilde{e}(x \otimes \iota_\infty)\tilde{f} + \tilde{c}(x \otimes \iota_\infty)\tilde{d} - a(x \otimes \iota_\infty)b = 0, \tag{8}$$

and the components of $\tilde{e}$ and $\tilde{f}$ lie E and F respectively.

Let $v_2 = \sum_{j=1}^{m} \tilde{e}_j \otimes \tilde{f}_j \in E \odot F$. Then by (8),

$$\varphi_{v_2 - v_0}(x) = \varphi_{v_2}(x) - \varphi_{v_0}(x) = -\tilde{c}(x \otimes \iota_\infty)\tilde{d},$$

so

$$\|v_2 - v_0\|_h = \|\varphi_{v_2 - v_0}\|_{cb} \leqslant \|\tilde{c}\| \cdot \|\tilde{d}\| \leqslant 3\varepsilon,$$

from above. Thus

$$\begin{aligned} \|v - v_2\|_h &\leqslant \|v - v_0\|_h + \|v_0 - v_2\|_h \\ &\leqslant \varepsilon^2 + 3\varepsilon \leqslant 4\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $v \in E \otimes_h F$, and we have proved that *(iii)* implies *(i)*. □

I am indebted to Professor R. R. Smith for showing me the proof of the following theorem and allowing me to reproduce it. The theorem is contained in [BS] as a remark.

Define $\varphi \otimes \psi$ on $X \odot Y$ by

$$\varphi \otimes \psi \left(\sum_{j=1}^{n} a_j \otimes b_j \right) = \sum_{j=1}^{n} \varphi(a_j) \otimes \psi(b_j).$$

Theorem 2.1.13 *Let H be a Hilbert space (not assumed to be separable). Let X, Y, E, and F be subspaces of $B(H)$. Suppose $\varphi : X \to E$ and $\psi : Y \to F$ are completely bounded. Then $\varphi \otimes \psi : X \otimes_h Y \to E \otimes_h F$ and $Ker(\varphi \otimes \psi) = Ker\ \varphi \otimes_h Y + X \otimes_h Ker\ \psi$.*

Proof

Now

$$\begin{aligned}
\left\|\sum_{j=1}^{n}\varphi(a_j)\otimes\psi(b_j)\right\|_h &\leqslant \left\|\sum_{j=1}^{n}\varphi(a_j)\varphi(a_j)^*\right\|^{\frac{1}{2}}\left\|\sum_{j=1}^{n}\psi(b_j)^*\psi(b_j)\right\|^{\frac{1}{2}} \\
&= \|(\varphi(a_1)\ldots\varphi(a_n))\|\left\|\begin{pmatrix}\psi(b_1)\\ \vdots \\ \psi(b_n)\end{pmatrix}\right\| \\
&\leqslant \|\varphi\|_{cb}\|\psi\|_{cb}\left\|\sum_{j=1}^{n}a_j\otimes b_j\right\|_h .
\end{aligned}$$

Hence $\varphi\otimes\psi$ may be extended to a bounded operator from $X\otimes_h Y$ to $E\otimes_h F$.

Clearly $Ker\ \varphi\otimes_h Y + X\otimes_h Ker\ \psi \subseteq Ker(\varphi\otimes\psi)$. Suppose then $u\in Ker(\varphi\otimes\psi)$ and use Lemma 2.1.8 to write $u=\sum_{j=1}^{\infty}a_j\otimes b_j$ with $\mathbb{N}=I_1\cup I_2$ where $a_j\in Ker\ \varphi$ for $j\in I_1$, $\{a_j : j\in I_1\}$ strongly independent, and $\{a_j : j\in I_2\}$ strongly independent over $Ker\ \varphi$ (discarding those a_j which are zero). Then

$$\sum_{j\in I_2}a_j\otimes b_j=\sum_{j\in\mathbb{N}}a_j\otimes b_j-\sum_{j\in I_1}a_j\otimes b_j\in Ker(\varphi\otimes\psi),$$

so

$$\sum_{j\in I_2}\varphi(a_j)\otimes\psi(b_j)=0.$$

We claim that $\{\varphi(a_j) : j\in I_2\}$ is strongly independent. If $\underline{\lambda}\in l^2$ and $\sum_{j\in I_2}\lambda_j\varphi(a_j)=0$ then $\varphi\left(\sum_{j\in I_2}\lambda_j a_j\right)=0$ so $\sum_{j\in I_2}\lambda_j a_j\in Ker\ \varphi$. However $\{a_j : j\in I_2\}$ is strongly independent over $Ker\ \varphi$ so we must have $\lambda_j=0$ for $j\in I_2$.

If $f\in B(H)^*$ then $(f(\psi(b_j)))\in l^2$ and $\sum_{j\in I_2}\varphi(a_j)f(\psi(b_j))=0$ so $f(\psi(b_j))=0$ for $j\in I_2$. However, f was arbitrary so $\psi(b_j)=0$ for $j\in I_2$. Thus

$$u=\sum_{j\in I_1}a_j\otimes b_j+\sum_{j\in I_2}a_j\otimes b_j,$$

with $a_j\in Ker\ \varphi$ for $j\in I_1$ and $b_j\in Ker\ \psi$ for $j\in I_2$ so

$$Ker(\varphi\otimes\psi)\subseteq Ker\ \varphi\otimes_h Y + X\otimes_h Ker\ \psi$$

□

2.2 The weak*-Haagerup tensor product

There is a natural identification of the dual of a tensor product with a space of bilinear forms, which dates back to the work of Grothendieck [G1, G2], given by

$$F(x, y) = \varphi(x \otimes y),$$

where F is a bilinear form and φ is in the dual of a tensor product. There is a result for the Haagerup tensor product of this form, first proved in an unpublished manuscript of Haagerup [H4], and later proved by Effros and Kishimoto [EK].

Theorem 2.2.1 *If X and Y are subspaces of C^*-algebras, then $CB(X \times Y, \mathbb{C})$ is isometrically isomorphic to $(X \otimes_h Y)^*$.*

The purpose of this section is to define a tensor product which gives an alternative characterisation of the dual of the Haagerup tensor product. This is known as the weak*-Haagerup tensor product and was introduced in a paper of Blecher and Paulsen [BP]. We define it and study some of its basic properties below, but first we need a result about the Haagerup tensor product, usually referred to as self-duality. This, in common with most of the initial results of this section, is proved both in Blecher's paper [B3] and in Effros and Ruan's [ER3]. Since these papers generally follow different paths and give alternative proofs, we refer to the paper whose approach we follow, always choosing the proof which is simplest in our context.

It will be necessary for us to extend the definition of the Haagerup tensor product from §2.1. Recall that if A and B are C*-algebras and $u \in A \odot B$ then we defined the Haagerup norm of u by

$$\begin{aligned} \|u\|_h &= inf\left\{ \left\| \sum_{j=1}^{n} a_j a_j^* \right\|^{\frac{1}{2}} \cdot \left\| \sum_{j=1}^{n} b_j^* b_j \right\|^{\frac{1}{2}} \right\} \\ &= inf\left\{ \|(a_1, \ldots, a_n)\| \cdot \left\| \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right\| \right\}, \end{aligned}$$

where the infimum runs over all representations $u = \sum_{j=1}^{n} a_j \otimes b_j$. We can extend this definition to any space where we can define norms of rows and columns over the space in a natural way. In particular, we can extend the definition to the dual of a C*-algebra. Suppose $v \in A^* \odot B^*$. Define the Haagerup norm of v by

$$\|v\|_h = inf\left\{\left\|\begin{pmatrix} f_1 & \cdots & f_n \\ 0 & \cdots & 0 \end{pmatrix}\right\| \cdot \left\|\begin{pmatrix} g_1 & 0 \\ \vdots & \vdots \\ g_n & 0 \end{pmatrix}\right\|\right\},$$

where the infimum runs over all representations $v = \sum_{j=1}^n f_j \otimes g_j$ and

$$\|(f_{ij})\| = sup\{\|(f_{ij}(x_{kl}))\|_{mn} : (x_{kl}) \in M_m(A),\ \|(x_{kl})\| \leqslant 1\}.$$

We denote by $A^* \otimes_h B^*$ the completion of $A^* \odot B^*$ in this norm.

Theorem 2.2.2 ([ER3]Theorem 3.2) *If A and B are subspaces of $B(H)$, then $A^* \otimes_h B^*$ is embedded isometrically in $(A \otimes_h B)^*$.*

Proof

We use the isometry $CB(X \times Y, \mathbb{C}) \cong (A \otimes_h B)^*$ given above. There is a natural embedding of $A^* \otimes_h B^*$ in $(A \otimes_h B)^*$ with $\sum_j f_j \otimes g_j \in A^* \odot B^*$ defining a map from $A \times B$ to $\mathbb{C}$ by

$$\left(\sum_j f_j \otimes g_j\right)(a, b) = \sum_j f_j(a) g_j(b).$$

We need to show that this map is completely bounded and that the embedding is isometric.

Suppose then that $(f_1, \ldots, f_m) \in M_{1,m}(A^*)$ and $(g_1, \ldots, g_m)^t \in M_{m,1}(B^*)$ so that

$$(f_1, \ldots, f_m) \otimes (g_1, \ldots, g_m)^t = \sum_{j=1}^m f_j \otimes g_j \in A^* \odot B^*.$$

Now for any C*-algebra A, $M_{1,m}(A^*)$ is isometrically isomorphic to $CB(A, M_{1,m}(\mathbb{C}))$ with $(f_1, \ldots, f_m) \in M_{1,m}(A^*)$ defining $\varphi \in CB(A, M_{1,m}(\mathbb{C}))$ by

$$\varphi(a) = (f_1(a), \ldots, f_m(a)), \qquad a \in A.$$

In the same way, $M_{m,1}(B^*)$ is isomorphic to $CB(B, M_{m,1}(\mathbb{C}))$. Thus, by the representation theorem for completely bounded maps (Theorem 1.2.3), we can write

$$(f_1, \ldots, f_m)(a) = S\pi_1(a)T, \qquad a \in A,$$

where $\pi_1 : A \to B(H_1)$ and $\mathbb{C}^m \xrightarrow{T} H_1 \xrightarrow{S} \mathbb{C}$, with $\|(f_1,\ldots,f_m)\|_{cb} = \|S\| \cdot \|T\|$, and

$$\begin{pmatrix} g_1 \\ \vdots \\ g_m \end{pmatrix}(b) = V\pi_2(b)W, \qquad b \in B,$$

where $\pi_2 : B \to B(H_2)$ and $\mathbb{C} \xrightarrow{W} H_2 \xrightarrow{V} \mathbb{C}^m$, with $\|(g_1,\ldots,g_m)^t\|_{cb} = \|V\| \cdot \|W\|$.

But by Theorem 2.2.1, $(A \otimes_h B)^* \cong CB(A \times B, \mathbb{C})$, so define $F : A \times B \to \mathbb{C}$ by

$$F(a,b) = S\pi_1(a)TV\pi_2(b)W \qquad a \in A, b \in B.$$

Then by Theorem 1.2.8,

$$\begin{aligned} \|F\|_{cb} &\leqslant \|S\| \cdot \|T\| \cdot \|V\| \cdot \|W\| \\ &= \|\sum_j f_j \otimes g_j\|_h. \end{aligned}$$

Conversely, suppose that $u \in A^* \odot B^*$ defines $F \in CB(A \times B, \mathbb{C})$ as above, and that $\|F\|_{cb} \leqslant 1$. We can write

$$F = \sum_{j=1}^n f_j \otimes g_j,$$

with $f_j \in A^*$ and $g_j \in B^*$. Let $A_0 \subseteq A$ and $B_0 \subseteq B$ be defined by

$$A_0 = \bigcap_{j=1}^n Ker\ f_j, \qquad B_0 = \bigcap_{j=1}^n Ker\ g_j,$$

and let $\rho : A \to A_1 = \frac{A}{A_0}$, $\sigma : B \to B_1 = \frac{B}{B_0}$ be the quotient maps. We can define a matrix structure on $\frac{A}{A_0}$ by using the identification $M_n(\frac{A}{A_0}) \cong \frac{M_n(A)}{M_n(A_0)}$. Then the quotient map is a complete contraction, that is $\|\rho\|_{cb} \leqslant 1$ (see [Ru] Theorem 4.2 for details). Now $f_j = \rho^*(\bar{f}_j)$ and $g_j = \sigma^*(\bar{g}_j)$ for suitable $\bar{f}_j, \bar{g}_j$ on the finite-dimensional spaces A_1, B_1 respectively, so

$$F = (\rho \otimes \sigma)^* \left(\sum_{j=1}^n \bar{f}_j \otimes \bar{g}_j \right).$$

Now $(\rho \otimes \sigma)^*$ is a complete isometry so if $\bar{F} : A_1 \otimes_h B_1 \to \mathbb{C}$ is given by $\bar{F} = \sum_{j=1}^n \bar{f}_j \otimes \bar{g}_j$ then $\|\bar{F}\|_{cb} \leqslant 1$. Then we can write

$$\bar{F}(a,b) = S\pi_1(a)T\pi_2(b)W \qquad a \in A, b \in B,$$

where $\pi_1 : A_1 \to B(H_1)$ and $\pi_2 : B_1 \to B(H_2)$ and $\mathbb{C} \xrightarrow{W} H_2 \xrightarrow{T} H_1 \xrightarrow{S} \mathbb{C}$ with $\|F\|_{cb} = \|S\| \cdot \|T\| \cdot \|W\| \leqslant 1$.

Let e_1, e_2 be the projections of H_1 and H_2 onto the finite-dimensional subspaces $H_1' = \pi_1(A_1)S^*\mathbb{C}$ and $H_2' = \pi_2(B_1)W^*\mathbb{C}$ respectively. Then

$$F(a,b) = S\pi_1(a)e_1Te_2\pi_2(b)W.$$

Let r be the rank of e_1Te_2. Then we can write $e_1Te_2 = V_1V_2$, where V_1, V_2 satisfy $H_2 \xrightarrow{V_2} \mathbb{C}^r \xrightarrow{V_1} H_1$ and $\|V_1\|, \|V_2\| \leqslant 1$. Let $(f_1, \ldots, f_r) \in M_{1,m}(\mathbb{C}) \otimes A_1^*$ be given by

$$(f_1, \ldots, f_m)(a) = S\pi_1(a)V_1, \qquad a \in A_1,$$

and $(g_1, \ldots, g_r)^t \in M_{m,1}(\mathbb{C}) \otimes B_1^*$ be given by

$$\begin{pmatrix} g_1 \\ \vdots \\ g_r \end{pmatrix}(b) = V\pi_2(b)W, \qquad b \in B_1.$$

Then $(\rho \otimes \sigma)^*(f_1, \ldots, f_r) \otimes (g_1, \ldots, g_r)^t$ gives a representation of u with

$$\|u\|_h \leqslant \|S\| \cdot \|V_1\| \cdot \|V_2\| \cdot \|W\| \leqslant 1.$$

□

To define $T(H) \otimes_h T(H)$, $T(H)$ must be an operator space. $T(H) \cong CB(K(H), \mathbb{C})$ and $CB(K(H), \mathbb{C})$ may be given a matricial norm arising from the isomorphism $M_n(CB(K(H), \mathbb{C}) \cong CB(M_{n,1}(K(H)), \mathbb{C}^n)$ [BP]. This is an L^∞-norm so by Ruan's theorem [Ru], $CB(K(H), \mathbb{C})$ is an operator space.

Definition 2.2.3 *If H is a Hilbert space then we define the weak*-Haagerup tensor product of $B(H)$ with itself, denoted $B(H) \otimes_{w^*h} B(H)$, to be the weak*-closure of $B(H) \odot B(H)$ in $(T(H) \otimes_h T(H))^*$.*

We will generalise this definition later. First we characterise the space $(T(H) \otimes_h T(H))^*$. We will need to define some more tensor norms, following [BP].

Definition 2.2.4 *(i) If X and Y are subspaces of C^*-algebras, then we define the tensor norm $\|\cdot\|_{max}$ for $(u_{ij}) \in M_n(X \odot Y)$ by*

$$\|(u_{ij})\|_{max} = sup\left\{ \left\| \left(\sum_k \psi(x_{ijk}, y_{ijk}) \right) \right\| : \psi \in CB(X \times Y, \mathbb{C}), \|\psi\|_{cb} \leqslant 1, \right.$$
$$\left. u_{ij} = \sum_k x_{ijk} \otimes y_{ijk},\ x_{ijk} \in X,\ y_{ijk} \in Y \right\},$$

where the supremum runs over all such ψ. We denote the completion of $X \odot Y$ in this norm by $X \otimes_{max} Y$.

(ii) If X and Y are subspaces of C^*-algebras, then we define the tensor norm $\|\cdot\|_{min}$ for $U = (u_{ij}) \in M_n(X \odot Y)$ by

$$\|U\|_{min} = sup\{\| < f \otimes g, U >_{mn} \| : f \in M_p(X^*), \|f\| \leqslant 1, \; g \in M_q(Y^*), \|g\| \leqslant 1\},$$

where $m = p + q$, $<,>$ is the natural dual action and the supremum runs over all m and n and such matrices of elements in the duals. Note that the norm is calculated in $M_{mn}(\mathbb{C})$. We denote the completion of $X \odot Y$ in this norm by $X \otimes_{min} Y$.

Proposition 2.2.5 ([BS] Proposition 2.1) *The following spaces are isometrically isomorphic:*

(i) $(T(H) \otimes_h T(H))^*$,

(ii) $(K(H) \otimes_{max} T(H))^*$,

(iii) $w^*CB(B(H), B(H))$, *the weak*-continuous completely bounded operators on* $B(H)$,

(iv) $CB(K(H), B(H))$,

(v) $CB(T(H), T(H))$.

The proof is through a series of lemmas, all of which may be extracted from [BP],[B3] and [ER3]

Lemma 2.2.6 ([BP] Proposition 5.4) *If X and Y are subspaces of C^*-algebras, then the following spaces are isometrically isomorphic:*

(i) $(X \otimes_{max} Y)^*$,

(ii) $CB(X \times Y, \mathbb{C})$,

(iii) $CB(X, Y^*)$,

(iv) $CB(Y, X^*)$.

Proof

We first show that $(X \otimes_{max} Y)^* \cong CB(X \times Y, \mathbb{C})$.

Given $F \in (X \otimes_{max} Y)^*$, $\|F\| \leqslant 1$, define $\psi \in B(X \times Y, \mathbb{C})$ by

$$\psi(x,y) = F(x \otimes y) \qquad x \in X,\ y \in Y.$$

Note that by Proposition 1.2.6, $\|F\|_{cb} \leqslant 1$. Then, if $(x_{ij}) \in M_n(X)$, $(y_{kl}) \in M_n(Y)$

$$\begin{aligned} |\psi((x_{ij}),(y_{ij}))| &= \left\| \left(\sum_{k=1}^{n} \psi(x_{ik}, y_{kj}) \right) \right\| \\ &= \left\| \left(\sum_{k=1}^{n} F(x_{ik} \otimes y_{kj}) \right) \right\| \\ &\leqslant \|F\|_{cb} \left\| \left(\sum_{k} x_{ik} \otimes y_{kj} \right) \right\|_{max} \\ &\leqslant \|(x_{ij})\| \cdot \|(y_{ij})\|. \end{aligned}$$

Thus ψ is completely bounded with $\|\psi\|_{cb} \leqslant \|F\|$.

Conversely, given $\psi \in CB(X \times Y, \mathbb{C})$ define $F \in (X \otimes_{max} Y)^*$ by

$$F\left(\sum_{r} x_r \otimes y_r \right) = \sum_{r} \psi(x_r, y_r).$$

Then

$$|F\left(\sum_{r} x_r \otimes y_r \right)| = |\sum_{r} \psi(x_r, y_r)| \leqslant \|\psi\| \cdot \| \sum_{r} x_r \otimes y_r \|_{max},$$

so $\|F\| \leqslant \|\psi\|_{cb}$. Hence $CB(X \times Y, \mathbb{C}) \cong (X \otimes_{max} Y)^*$ isometrically.

Now suppose $\psi \in CB(X \times Y, \mathbb{C})$ with $\|\psi\|_{cb} \leqslant 1$. Define $\varphi \in CB(X, Y^*)$ by

$$\varphi(x)y = \psi(x,y) \qquad x \in X,\ y \in Y.$$

Note that if $(x_{ij}) \in M_n(X)$, $(y_{ij}) \in M_n(Y)$, then

$$\begin{aligned} \|\varphi_n((x_{ij}))(y_{ij})\| &= \|(\varphi(x_{ij}))\,(y_{ij})\| \\ &= \left\| \left(\sum_{k=1}^{n} \varphi(x_{ik}) y_{kj} \right) \right\| \\ &= \left\| \left(\sum_{k=1}^{n} \psi(x_{ik}, y_{kj}) \right) \right\| \\ &= \|\psi_n\left((x_{ij}),(y_{ij})\right)\|, \end{aligned}$$

and thus that $\|\varphi_n\| = \|\psi_n\|$ for all $n \in \mathbb{N}$. Hence $\|\varphi\|_{cb} = \|\psi\|_{cb}$. Similarly, given $\varphi \in CB(X, Y^*)$, define $\psi \in CB(X \times Y, \mathbb{C})$ by the above and then $\|\psi\|_{cb} = \|\varphi\|_{cb}$. Thus

$$CB(X \times Y, \mathbb{C}) \cong CB(X, Y^*)$$

isometrically.

Similarly, $CB(Y, X^*) \cong CB(X \times Y, \mathbb{C})$ with the isometric isomorphism given by $\varphi(y)x = \psi(x, y)$. $\square$

Before the next lemma we must define Hilbert row and column space. For a Hilbert space H, we define the corresponding Hilbert row space, denoted H_c to be $B(\mathbb{C}, H)$, and Hilbert column space, denoted H_r to be $B(H, \mathbb{C})$. Given a rank one projection $e \in B(H)$, we may identify H_c with $B(H)e$ and H_r with $eB(H)$ [Rob].

Lemma 2.2.7 ([B3] Theorem 2.2) *If H and K are Hilbert spaces, then*

$$CB(H_c, K_c) \cong B(H, K) \text{ and } CB(H_r, K_r) \cong B(K, H),$$

isometrically. In particular, $(H_c)^* \cong H_r$ and $(H_r)^* \cong H_c$.

Proof

Suppose $\varphi \in CB(H_c, K_c)$ with $\|\varphi\|_{cb} \leqslant 1$. Then given $\varepsilon > 0$ there exists n and $(\xi_{ij}) \in M_n(H_c)$ with $\|(\xi_{ij})\| \leqslant 1$ and

$$\begin{aligned} \|\varphi\|_{cb} &< \|\varphi_n(\xi_{ij})\| + \varepsilon \\ &= \|(\varphi(\xi_{ij}))\| + \varepsilon \end{aligned}$$

Now the linear span of $\{\xi_{ij} : 1 \leqslant i, j \leqslant n\}$ is a finite dimensional subspace of H_c and so is isomorphic to $\mathbb{C}^m$ for some $m \in \mathbb{N}$. Also, the linear span of $\{\varphi(\xi_{ij}) : 1 \leqslant i, j \leqslant n\}$ is isomorphic to $\mathbb{C}^k$ for some $k \in \mathbb{N}$.

Let $\tilde{\varphi}$ be the restriction of φ to the linear span of $\{\xi_{ij}\}$. Considering $\mathbb{C}^m$ as the leftmost column of an $m \times m$ matrix it is easy to see that any map φ from $\mathbb{C}^m$ to $\mathbb{C}^k$ arises as a multiplication on the left by an element x_φ of $M_{k,m}(\mathbb{C})$, and then the representation theorem for completely bounded maps tells us that this operator has norm equal to the norm of the matrix. That is, $CB(\mathbb{C}^m, \mathbb{C}^k) \cong M_{k,m}(\mathbb{C})$ isometrically. Hence

$$\|\varphi\|_{cb} < \|x_\varphi\| + \varepsilon.$$

Conversely,

$$\begin{aligned} \|\varphi\|_{cb} &> \|\varphi_n(\xi_{ij})\| - \varepsilon \\ &= \|x_\varphi\| - \varepsilon. \end{aligned}$$

Thus $CB(H_c, K_c) \cong B(H, K)$.

Now, since $CB(X^*, \mathbb{C}) \cong B(X^*, \mathbb{C})$ isometrically, for all Banach spaces X (Proposition 1.2.6),

$$(H_c)^* \cong CB(H_c, \mathbb{C}) \cong B(H, \mathbb{C}) = H_r,$$

using the isometric isomorphism given in the first part. The other parts are proved by a similar reduction to the finite-dimensional case. □

Lemma 2.2.8 ([BP], Theorem 2.11) *If X is a subspace of a C^*-algebra, then the canonical embedding of X into $X^{**} = CB(CB(X, \mathbb{C}), \mathbb{C})$ is a complete isometry.*

Proof

We can assume that $X \subseteq B(H)$ for some Hilbert space H. Let $\mathcal{F}$ be the family of finite dimensional subspaces F of H. For $F \in \mathcal{F}$ write φ_F for the compression of a map φ from $B(H)$ to $B(F)$.

If $\wedge : X \to X^{**}$ is the canonical embedding and $(x_{ij}) \in M_n(X)$ then

$$\begin{aligned} \|(\hat{x_{ij}})\|_n &= sup\{\|(\hat{x_{ij}}(f_{kl}))\| : (f_{kl}) \in M_m(X^*), \|(f_{kl})\| \leqslant 1\} \\ &= sup\{\|(f_{kl}(x_{ij}))\| : (f_{kl}) \in M_m(X^*), \|(f_{kl})\| \leqslant 1\} \\ &= sup\{\|T_n(x_{ij})\| : T \in CB(X, M_m(\mathbb{C})), \|T\|_{cb} \leqslant 1\}. \end{aligned}$$

This last quantity is less than or equal to $\|(x_{ij})\|$, and since

$$\|(x_{ij})\| = sup\{\|(\varphi_F)_n(x_{ij})\| : F \in \mathcal{F}\},$$

and each φ_F may be regarded as a completely bounded map into some $M_m(\mathbb{C})$ (with $m = dim\ F$), the result follows. □

Lemma 2.2.9 ([BP], Theorem 5.6) *If X and Y are subspaces of C^*-algebras, then $X^* \otimes_{min} Y^*$ embeds isometrically in $(X \otimes_{max} Y)^*$.*

Proof

Consider the diagram

$$\begin{tikzcd}
X^* \otimes_{min} Y^* \arrow[rr] \arrow[d] & & (X \otimes_{max} Y)^* \arrow[d] \\
CB(X^{**}, Y^*) \arrow[dr] & & CB(X, Y^*) \arrow[dl] \\
& CB(X^{**}, Y^{***}) &
\end{tikzcd}$$

where all the maps are the canonical embeddings, the embedding of $CB(X, Y^*)$ in $CB(X^{**}, Y^{***})$ simply uses the second Banach space adjoint, and so is isometric.

Given $\sum_j f_j \otimes g_j \in X^* \otimes_{min} Y^*$, $F \in M_n(X^{**})$ and $y \in M_m(Y)$,

$$\begin{aligned}
\left\| \left(\sum_j f_j \otimes g_j \right) \otimes \iota_n(F)(y) \right\| &= \left\| \sum_j [F(f_j \otimes \iota_n)][(g_j \otimes \iota_n)(y)] \right\| \\
&= \left\| F \otimes \hat{y} \left[\left(\sum_j f_j \otimes g_j \right) \otimes \iota_n \right] \right\|,
\end{aligned}$$

so by the previous lemma, the embedding of $X^* \otimes_{min} Y^*$ in $CB(X^{**}, Y^*)$ is isometric.

We have already seen that the embedding of $(X \otimes_{max} Y)^*$ in $CB(X, Y^*)$ is isometric and the discussion after Theorem 2.1.4 shows that the embedding of $CB(X^{**}, Y^*)$ in $CB(X^{**}, Y^{***})$ is isometric. Thus, if the diagram commutes, the embedding of $X^* \otimes_{min} Y^*$ in $(X \otimes Y)^*$ must be isometric.

However, if $\sum_{j=1}^n f_j \otimes g_j \in X^* \odot Y^*$ then composing clockwise, we first obtain the map $\sum_{j=1}^n f_j(\cdot) g_j \in CB(X, Y^*)$ and then the map $F \mapsto \sum_{j=1}^n F(f_j)\hat{g}_j \in CB(X^{**}, Y^{***})$. Composing anticlockwise, we first obtain the map $F \mapsto \sum_{j=1}^n F(f_j) g_j \in CB(X^{**}, Y^*)$ and then the required map in $CB(X^{**}, Y^{***})$. □

Lemma 2.2.10 *If H is a Hilbert space and X is a subspace of a C^*-algebra, then*

$$H_r \otimes_{max} X \cong H_r \otimes_h X \text{ and } H_c \otimes_{max} X \cong X \otimes_h H_c$$

isometrically.

Proof

From Lemma 2.2.6, $(H_r \otimes_{max} X)^* \cong CB(H_r, X^*)$ and from Theorem 2.2.1, $(H_r \otimes_h X)^* \cong CB(H_r \times X, \mathbb{C})$. Given $\varphi : H_r \to X^*$, define $F : H_r \times X \to \mathbb{C}$ by

$$F(y, x) = \varphi(y)x \qquad x \in X, y \in H_r.$$

Then if $(x_{ij}) \in M_n(X)$, $(y_{ij}) \in M_n(H_r)$,

$$\begin{aligned} \|F_n((y_{ij}), (x_{ij}))\| &= \left\| \left(\sum_{k=1}^{n} F(y_{ik}, x_{kj}) \right) \right\| \\ &= \left\| \left(\sum_{k=1}^{n} \varphi(y_{ik}) x_{kj} \right) \right\| \\ &= \|\varphi_n((y_{ij}))(x_{ij})\|. \end{aligned}$$

Hence $\|F_n\| = \|\varphi_n\|$ for all $n \in \mathbb{N}$ and so $\|F\|_{cb} = \|\varphi\|_{cb}$. Thus

$$(H_r \otimes_{max} X)^* \cong (H_r \otimes X)^*.$$

The other case is similar. □

Lemma 2.2.11 ([ER3], Corollary 4.4) *If H is a Hilbert space, then*

$$H_r \otimes_{min} H_c \cong H_c \otimes_h H_r \cong K(H)$$

isometrically.

Proof

From Lemmas 2.2.7 and 2.2.9, $H_r \otimes_{min} H_c$ embeds isometrically in $(H_c \otimes_{max} H_r)^*$. But, by Lemma 2.2.6, $(H_c \otimes_{max} H_r)^* \cong (H_c \otimes_h H_r)^*$, and then by Theorem 2.2.2, $(H_c \otimes_h H_r)^*$ embeds isometrically in $H_c \otimes_h H_r$. Hence $H_r \otimes_{min} H_c \cong H_c \otimes_h H_r$.

Further,

$$\begin{aligned} (H_c \otimes_h H_r)^{**} &\cong (H_r \otimes_h H_c)^* && (2.2.2) \\ &\cong CB(H_r \times H_r, \mathbb{C}) && (2.2.1) \\ &\cong CB(H_r, H_r) && (2.2.7) \\ &\cong B(H). && (2.2.7) \end{aligned}$$

Therefore, $H_c \otimes_h H_r \cong K(H)$. □

Lemma 2.2.12 ([BP]) *If X, Y and Z are subspaces of C^*-algebras then*

$$\begin{aligned} X \otimes_{max} (Y \otimes_{max} Z) &\cong (X \otimes_{max} Y) \otimes_{max} Z, \\ \text{and } X \otimes_h (Y \otimes_h Z) &\cong (X \otimes_h Y) \otimes_h Z. \end{aligned}$$

Also, $X \otimes_{max} Y \cong Y \otimes_{max} X$.

Proof

The last statement follows from the symmetry in the definition of the max norm. We now prove the first statement; the second is similar. We use Lemma 2.2.6 repeatedly.

$$\begin{aligned} (X \otimes_{max} (Y \otimes_{max} Z))^* &\cong CB(X, (Y \otimes_{max} Z)^*) \\ &\cong CB(X, CB(Y, Z^*)) \\ &\cong CB(Z, CB(X, Y^*)) \\ &\cong CB(Z, (X \otimes_{max} Y)^*) \\ &\cong ((X \otimes_{max} Y) \otimes_{max} Z)^*, \end{aligned}$$

where $\varphi \in CB(X, CB(Y, Z^*))$ and $\psi \in CB(Z, CB(X, Y^*))$ are linked by $\varphi(x)y(z) = \psi(z)x(y)$. □

Note that the Haagerup norm is not commutative, that is $A \otimes_h B$ is not isometrically isomorphic to $B \otimes_h A$. This is because in the Haagerup norm xx^* occurs in the first variable and y^*y in the second. We will see an example of this in Chapter 3.

We can now prove Proposition 2.2.5

Proof of Proposition 2.2.5

By Lemma 2.2.6, $(ii) = (iv) = (v)$.

The proof that $(iii) = (iv)$ is contained in a remark after Theorem 2.1.10.

All that remains is to prove $(i) = (ii)$, namely that

$$(T(H) \otimes_h T(H))^* \cong (K(H) \otimes_{max} T(H))^*.$$

First we note that

$$\begin{aligned} T(H) = K(H)^* &\cong (H_r \otimes_{min} H_c)^* \quad (2.2.7) \\ &\cong H_r \otimes_{max} H_c \qquad (2.2.9). \end{aligned}$$

Then

$$\begin{aligned}
K(H)\otimes_{max} T(H) &\cong K(H)\otimes_{max}(H_r\otimes_{max} H_c) \\
&\cong H_r\otimes_{max} K(H)\otimes_{max} H_c && (2.2.12)\\
&\cong H_r\otimes_{max}(H_c\otimes_h H_r)\otimes_{max} H_c && (2.2.7)\\
&\cong H_r\otimes_h (H_c\otimes_h H_r)\otimes_{max} H_c && (2.2.10)\\
&\cong H_r\otimes_h (H_c\otimes_h H_r)\otimes_h H_c && (2.2.10)\\
&\cong (H_r\otimes_h H_c)\otimes_h (H_r\otimes_h H_c) && (2.2.12)\\
&\cong (H_r\otimes_{max} H_c)\otimes_h (H_r\otimes_{max} H_c) && (2.2.10)\\
&\cong T(H)\otimes_h T(H). && \square
\end{aligned}$$

The proposition now shows that the weak*-topologies generated by $(T(H)\otimes_h T(H))^*$ and $(K(H)\otimes_{max} T(H))^*$ are the same, which we will refer to as the weak*-topology on any of these spaces.

Theorem 2.2.13 ([BS] Theorem 2.2) *If $\varphi : K(H) \to B(H)$ is a completely bounded operator then there exist sequences $\{a_j\}$ and $\{b_j\}$ in $B(H)$ such that $\sum_j a_j a_j^*$, $\sum_j b_j^* b_j$ converge strongly in $B(H)$,*

$$\|\sum_j a_j a_j^*\| \cdot \|\sum_j b_j^* b_j\| = \|\varphi\|_{cb}^2$$

and,

$$\varphi(x) = \sum_j a_j x b_j \qquad x \in K(H),$$

and the convergence of the partial sums of $\sum_j a_j \cdot b_j$ in $CB(K(H), B(H))$ is in the weak-topology on $CB(K(H), B(H))$.*

Proof

Except for the last statement, this is Theorem 2.1.4.

As we have seen in §2.1, for any $x \in K(H)$, the sum $\sum_j a_j x b_j$ converges in the weak operator topology. That is if $x \in K(H)$ and if $\xi, \eta \in H$ then

$$< (\sum_j a_j x b_j)\xi, \eta >= \sum_j < a_j x b_j \xi, \eta >$$

converges, and so the partial sums of $\sum_j a_j \cdot b_j$ converge when applied to the elementary tensors $x \otimes (\xi \otimes \eta)$ from the predual $K(H) \otimes_{max} T(H)$ by the above formula. However these elementary tensors span a norm dense set, since the rank one operators are of the form $\xi \otimes \eta$ and are dense in $T(H)$ (see §1.3). The partial sums of $\sum_j a_j \cdot b_j$ are uniformly bounded by $\left\|\sum_j a_j a_j^*\right\|^{\frac{1}{2}} \cdot \left\|\sum_j b_j^* b_j\right\|^{\frac{1}{2}}$, as we have seen in §2.1, . Thus it follows that $\sum_j a_j \cdot b_j$ converges in the weak*-topology. □

Corollary 2.2.14 ([BS], Corollary 2.3) *The space $B(H) \otimes_{w^*h} B(H)$ coincides with the five spaces in Proposition 2.2.5.*

Proof

We need to show that the image of $B(H) \odot B(H)$ under the map $v \mapsto \varphi_v$ is weak*-dense in $CB(K(H), B(H))$. However this follows from the above theorem. □

Thus we can identify with each $v \in B(H) \otimes_{w^*h} B(H)$ a representation $v = \sum_j a_j \otimes b_j$, which converges in the weak*-topology to v and which satisfies

$$\left\|\sum_j a_j a_j^*\right\|^{\frac{1}{2}} \cdot \left\|\sum_j b_j^* b_j\right\|^{\frac{1}{2}} = \|\varphi_v\|_{cb}^2.$$

We shall call this a weak*-representation of v and we remark that it is not in general unique.

Let X and Y be weak*-closed subspaces of $B(H)$. Define $X \otimes_{w^*h} Y$ to be the weak*-closure in $B(H) \otimes_{w^*h} B(H)$ of $X \odot Y$. It follows from Theorem 2.1.4 that $X \otimes_h Y$ is isometrically embedded in $X \otimes_{w^*h} Y$.

We can define the slice maps as for the Haagerup tensor product. If $\psi \in X^*$ is a normal linear functional, then $R_\psi : X \otimes_{w^*h} Y \to Y$ is given by

$$R_\psi(\sum_j a_j \otimes b_j) = \sum_j \psi(a_j) b_j, \qquad \sum_j a_j \otimes b_j \in X \otimes_{w^*h} Y.$$

As we saw when we defined the slice maps on the Haagerup tensor product, the fact that all linear functionals are completely bounded means that $\sum_j |\psi(a_j)|^2 < \infty$. Then by Lemma 2.1.6 R_ψ is bounded and $\sum_j \psi(a_j) b_j$ converges in norm.

Theorem 2.2.15 ([BS], Theorem 3.1) *Let X and Y be weak*-closed subspaces of $B(H)$. An element $v \in B(H) \otimes_{w^*h} B(H)$ lies in $X \otimes_{w^*h} Y$ if and only if v has a* *weak*-representation* $v = \sum_j a_j \otimes b_j$ *with* $a_j \in X$ *and* $b_j \in Y$. *Moreover*

$$\|v\|_{w^*h} = inf \left\{ \left\| \sum_j a_j a_j^* \right\|^{\frac{1}{2}} \cdot \left\| \sum_j b_j^* b_j \right\|^{\frac{1}{2}} \right\},$$

where the infimum, which is attained, is taken over all such weak-representations of* v. *In this case* $v \in X \otimes_h Y$ *if and only if* v *has a weak*-representation which converges uniformly, and then the infimum above may be taken over all such representations.*

Proof

If $v \in X \otimes_{w^*h} Y$ then there is a net in $X \odot Y$ which converges to v in the weak*-topology. Since the slice maps are weak*-continuous it follows that v has the given weak*-representation.

Conversely if v has the given weak*-representation then the partial sums of $\sum_j a_j \otimes b_j$ converge in the weak*-topology to v, and so $v \in X \otimes_{w^*h} Y$.

If v has a uniformly convergent weak*-representation then clearly $v \in X \otimes_h Y$.

Conversely, suppose that $v \in X \otimes_h Y$ with $\|v\|_h \leqslant 1$. Then given $\varepsilon > 0$ we may choose $v_1 = \sum_{j=1}^{n_1} a_j \otimes b_j$ with $a_j \in X$, $b_j \in Y$, $\|v - v_1\|_h < \varepsilon$, and $\|\sum_{j=1}^{n_1} a_j a_j^*\| = \|\sum_{j=1}^{n_1} b_j^* b_j\| < 1$. Now choose $v_2 = \sum_{j=n_1+1}^{n_2} a_j \otimes b_j$, with $a_j \in X$, $b_j \in Y$, $\|v - v_1 - v_2\| < \frac{\varepsilon}{2}$, and $\|\sum_{j=n_1+1}^{n_2} a_j a_j^*\| = \|\sum_{j=n_1+1}^{n_2} b_j^* b_j\| < \varepsilon$. Choose $v_3 = \sum_{j=n_2+1}^{n_3} a_j \otimes b_j$ with $a_j \in X$, $b_j \in Y$ and $\|v - v_1 - v_2 - v_3\|_h < \frac{\varepsilon}{4}$, and so on. Then we will have that $\|\sum_j a_j a_j^*\|$ and $\|\sum_j b_j^* b_j\|$ are each less than $1 + 2\varepsilon$, and that $\sum_j a_j \otimes b_j$ converges in the weak*-topology as in Theorem 2.2.11. Since $\sum_j v_j$ converges uniformly to v, it follows that v has weak*-representation $\sum_j a_j \otimes b_j$. The remaining assertion follows from this construction. □

We now need to extend the definition from that used by Blecher and Smith so that we can define the weak*-Haagerup tensor product of subspaces of C*-algebras.

Definition 2.2.16 *Suppose that* A *and* B *are* C^**-algebras with representations on Hilbert spaces* H *and* K *respectively, and that* X *and* Y *are closed subspaces of* A

and B respectively. We define the weak-Haagerup tensor product of X and Y by*

$$X \otimes_{w^*h} Y = \left\{ \begin{array}{ll} u \in A^{**} \otimes_{w^*h} B^{**} : & u = \sum_j a_j \otimes b_j \text{ with } a_j \in X,\ b_j \in Y, \\ & \sum_j a_j a_j^* \text{ converges strongly in } B(H), \\ & \sum_j b_j^* b_j \text{ converges strongly in } B(K) \end{array} \right\}.$$

The norm on the elements of $X \otimes_{w^*h} Y$ is the Haagerup norm.

It is easiest to think of this as formal infinite sums $\sum a_j \otimes b_j$ where the sums $\sum a_j a_j^*$ and $\sum b_j^* b_j$ converge strongly. One might expect that with this strong convergence, $K(H) \otimes_{w^*h} K(H)$ would be all of $B(H) \otimes_{w^*h} B(H)$ but we shall see later that this is not so.

From the discussion of the definition of completely bounded operators in Chapter 1, it is easy to see that $CB(B(H))$ is a Banach algebra, and hence that the weak*-continuous completely bounded operators on $B(H)$ are a Banach algebra. Then using Proposition 2.2.5, we have that $B(H) \otimes_{w^*h} B(H)$ is a Banach algebra.

Lemma 2.2.17 *If A and B are C^*-algebras, then $A \otimes_{w^*h} B$ is a Banach algebra.*

Proof

We assume that A and B are both represented on the same Hilbert space H, taking a direct sum if necessary.

Suppose $u_n \in A \otimes_{w^*h} B$ with $\sum_n \|u_n\|_{w^*h} < \infty$. There is a representation $u_n = \sum_j a_j^{(n)} \otimes b_j^{(n)}$ with

$$\|u_n\|_{w^*h} = \left\| \sum_j a_j^{(n)} a_j^{(n)*} \right\| = \left\| \sum_j b_j^{(n)*} b_j^{(n)} \right\|.$$

If $\xi \in H$, with $\|\xi\| \leqslant 1$ then

$$\begin{aligned} \left\| \sum_{j,n} a_j^{(n)} a_j^{(n)*} \xi \right\| &\leqslant \sum_n \left\| \sum_j a_j^{(n)} a_j^{(n)*} \xi \right\| \\ &\leqslant \sum_n \|u_n\|_{w^*h} \\ &< \infty. \end{aligned}$$

Similarly $\sum_{j,n} b_j^{(n)*} b_j^{(n)}$ converges strongly so $\sum_n u_n \in A \otimes_{w^*h} B$, and

$$\left\| \sum_{n=1}^{\infty} u_n - \sum_{j=1}^{m} u_n \right\|_{w^*h} \leqslant \sum_{n=m+1}^{\infty} \|u_n\|_{w^*h} \to 0 \text{ as } m \to \infty.$$

Thus $A \otimes_{w^*h} B$ is a Banach space.

If $\sum_j a_j \otimes b_j,\ \sum_k c_k \otimes d_k \in A \otimes_{w^*h} B$ then

$$\left(\sum_j a_j \otimes b_j\right)\left(\sum_k c_k \otimes d_k\right) = \sum_{j,k} a_j c_k \otimes d_k b_j,$$

with $a_j c_k \in A$ and $d_k b_j \in B$.

If $\xi \in H$ with $\|\xi\| \leqslant 1$ then

$$\begin{aligned}\left\langle \sum_{j,k} a_j c_k (a_j c_k)^* \xi, \xi \right\rangle &= \sum_j \left\langle a_j \left(\sum_k c_k c_k^*\right) a_j^* \xi, \xi \right\rangle \\ &\leqslant \left\|\sum_k c_k c_k^*\right\| \sum_j < a_j a_j^* \xi, \xi >,\end{aligned}$$

which converges in j and k by the strong convergence of the sums $\sum_j a_j a_j^*$ and $\sum_k c_k c_k^*$. Hence $\sum_{j,k} a_j c_k (a_j c_k)^*$ is strongly convergent and similarly, $\sum_{j,k}(d_k b_j)^* d_k b_j$ is strongly convergent. Further, the Banach algebra product inequality $\|u \cdot v\|_{w^*h} \leqslant \|u\|_{w^*h}\|v\|_{w^*h}$ follows directly from these calculations.

Thus $A \otimes_{w^*h} B$ is a Banach algebra. □

3 Ideals of Completely Bounded Operators

In this chapter we endeavour to answer the main question asked in the introduction: namely, if we know that a completely bounded operator is compact what can we say about its representation? This naturally leads us into considering the ideals of the algebra of completely bounded operators.

In section 1 we identify completely the ideals of $B(H) \otimes_h B(H)$. We show that the obvious candidates for ideals of $A \otimes_h A$, those of the form $J_1 \otimes_h J_2$ for ideals J_1, J_2 of A, are indeed ideals. Restricting attention to $B(H) \otimes_h B(H)$, we then show that there is only one other ideal which arises as a sum of two of this form. This ideal is then maximal. The proof of this is based on ideas of Professor R. R. Smith. The results in this section rely heavily on the existence of minimal projections in $B(H)$ and the ideal structure of $B(H)$, so do not readily generalise to other algebras. This work, and further results will appear in a joint paper with R. R. Smith and A. M. Sinclair, currently being prepared.

In section 2 we turn our attention to $B(H) \otimes_{w^*h} B(H)$. We begin by extending the definition of the weak*-Haagerup tensor product to C*-algebras, so that we may define $K(H) \otimes_{w^*h} K(H)$. We then show that the obvious basic candidates for ideals are again distinct ideals. We construct a compact completely bounded operator which is not approximable in completely bounded operator norm by finite ranks, thereby showing that the set of completely bounded compact operators and the completely bounded closure of the finite ranks are distinct ideals of $B(H) \otimes_{w^*h} B(H)$ and negatively answering our original question. We do, however, show that any compact completely bounded operator has a representation as an element of $K(H) \otimes_{w^*h} K(H)$.

Throughout this chapter we make constant use of the isomorphism

$$B(H) \otimes_{w^*h} B(H) \cong CB(K(H), B(H)),$$

given by $u = \sum_j a_j \otimes b_j$ defining $\varphi_u(x) = \sum_j a_j x b_j$, as detailed in the last chapter. When proving results about the ideals of the tensor product, we often calculate with completely bounded operators for simplicity. For example, the isometry observed in §2.1 of $B(H) \otimes_h B(H)$ in $CB(K(H), B(H))$ tells us that $B(H) \otimes_h B(H)$ is isometrically contained in $B(H) \otimes_{w^*h} B(H)$.

3.1 Ideals of $B(H) \otimes_h B(H)$

One would expect the ideal structure of $B(H) \otimes_h B(H)$ to be relatively straightforward because the representations of elements of the Haagerup tensor converge in norm. If A is a C*-algebra, then the obvious basic candidates for closed ideals of $A \otimes_h A$ are of the form $J_1 \otimes_h J_2$ where J_1 and J_2 are closed ideals of A.

Lemma 3.1.1 *If J_1 and J_2 are closed ideals of a C*-algebra, then $J_1 \otimes_h J_2$ is a closed ideal of $A \otimes_h A$, of $A \otimes_h J_2$ and of $J_1 \otimes_h A$.*

Proof

If $\sum_j a_j \otimes b_j \in J_1 \odot J_2$ and $\sum_i c_i \otimes d_i$, $\sum_k f_k \otimes g_k \in A \odot A$ then

$$\left(\sum_i c_i \otimes d_i\right)\left(\sum_j a_j \otimes b_j\right)\left(\sum_k f_k \otimes g_k\right) = \sum_{i,j,k} c_i a_j f_k \otimes g_k b_j d_i \quad \in \quad J_1 \odot J_2$$

since J_1 and J_2 are ideals of A.

Thus $J_1 \odot J_2$ is an ideal in $A \odot A$. Now the Haagerup tensor product is injective ([PS], Theorem 4.4), that is if $X_1 \subseteq X$ and $Y_1 \subseteq Y$ then $X_1 \otimes_h Y_1 \subseteq X \otimes_h Y$ isometrically. Hence, completing in the Haagerup norm, $J_1 \otimes_h J_2$ is a closed ideal in $A \otimes_h A$.

The other cases are identical. □

If A is a C*-algebra, then for $a, b \in A$ define the operator $L_a R_b$ on A to be the map $x \mapsto axb$. Such an operator is often called an elementary operator. Vala [V1] calls an element a of a C*-algebra A compact if the mapping $a \mapsto axa$ is a compact operator on A. Alexander [A] , following Vala, shows that if X is a Banach space and T is a compact operator on X, then $S \mapsto TST$ is a compact operator on $B(X)$. Hence Vala's definition of compactness agrees with the usual notion in this setting. The following theorem is due to Vala [V1] (part (i)) and Akemann & Wright [AW] (part (ii)). For a more detailed discussion of the compactness of elementary operators see the papers of Vala, Mathieu and Akemann & Wright[V1, V2, M1, M2, AW]. There is also a good account in §33 of Bonsall and Duncan's book [BD].

Theorem 3.1.2 *Let H be a Hilbert space,* and a, b be non-zero elements of $B(H)$.

(i) The operator $L_a R_b$ on $B(H)$ is compact if and only if a and b are compact operators on H.

(ii) The operator $L_a R_b$ on $B(H)$ is weakly compact if and only if at least one a and b is a compact operator on H.

The proof that $L_a R_b$ is compact when $a, b \in K(H)$ is an application of Ascoli-Arzelà, considering the compactness of the appropriate ranges. The converse is proved by noting that if $\xi, \eta \in H$ then $L_a R_b(\xi \otimes \eta) = a\eta \otimes b^*\xi$ so that $L_a R_b$ being compact guarantees the compactness of a and b^*, and of course if b^* is compact then so is b. The weakly compact case relies on noting that if φ is a weakly compact operator on $B(H)$ then the range of φ is contained in $K(H)$. Then considering whether or not the appropriate ranges have closed infinite-dimensional subspaces gives the result.

Corollary 3.1.3 *If $u \in B(H) \otimes_h K(H)$ or $u \in K(H) \otimes_h B(H)$ then φ_u is a weakly compact operator on $K(H)$. If $u \in K(H) \otimes_h K(H)$ then φ_u is a compact operator on $K(H)$.*

Proof

Suppose $u \in B(H) \otimes_h K(H)$. Then, given $\varepsilon > 0$, there exist sequences $\{a_j\}$, $\{b_j\}$ in $B(H)$ and $K(H)$ respectively such that

$$\left\| u - \sum_{j=1}^{n} a_j \otimes b_j \right\|_h < \varepsilon.$$

Now, by the above theorem, $a_j \otimes b_j$ is a weakly compact operator for $j = 1, \ldots, n$. However, the set of weakly compact operators form an ideal of $B(K(H))$ (Theorem 1.3.7) so $\sum_{j=1}^{n} a_j \otimes b_j$ is a weakly compact operator. Since the set of weakly compact operators is closed, φ_u is weakly compact. The other cases are similar. □

Since $B(H) \otimes_h B(H)$ clearly contains non-compact operators on $B(H)$, this shows that $B(H) \otimes_h B(H)$, $B(H) \otimes_h K(H)$ and $K(H) \otimes_h K(H)$ are all distinct. It remains to show that $K(H) \otimes_h B(H)$ and $B(H) \otimes_h K(H)$ are distinct.

Suppose then that $\mathbf{1} \otimes y \in K(H) \otimes_h B(H)$ for some $y \in K(H)$. There certainly exists $\psi \in B(H)^*$ with $\psi(y) \neq 0$, and then $0 \neq \psi(y)\mathbf{1} \in B(H)$. But by Theorem 2.1.12, $R_\psi(\mathbf{1} \otimes y) \in K(H)$ for all $\psi \in B(H)^*$, which gives us a contradiction. Similarly, $x \otimes \mathbf{1} \notin B(H) \otimes_h K(H)$ for any $x \in K(H)$.

Clearly if J_1 and J_2 are ideals of an algebra A, then $J_1 + J_2$ is also an ideal of A. Hence $[\ K(H) \otimes_h B(H)] + [\ B(H) \otimes_h K(H)]$ is an ideal of $B(H) \otimes_h B(H)$.

Proposition 3.1.4 *The ideal* $[B(H) \otimes_h K(H)] + [K(H) \otimes_h B(H)]$ *is closed in* $B(H) \otimes_h B(H)$.

Proof

Let $q : B(H) \to \frac{B(H)}{K(H)}$ be the completely bounded quotient map. Then by Theorem 2.1.13,

$$Ker(q \otimes q) = [B(H) \otimes_h K(H)] + [K(H) \otimes_h B(H)],$$

which is therefore closed. $\square$

Further, the ideal is clearly distinct from $B(H) \otimes_h K(H)$ and $K(H) \otimes_h B(H)$ since if $x \in K(H)$ then $\mathbf{1} \otimes x$ and $x \otimes \mathbf{1}$ are elements of it. Also, the ideal contains only weakly compact operators on $B(H)$, so it is also distinct from $B(H) \otimes_h B(H)$.

We now aim to show that the ideals already noted are all the closed ideals of $B(H) \otimes_h B(H)$.

Theorem 3.1.5 *Let H be a Hilbert space. Then $K(H) \otimes_h K(H)$ is a maximal closed ideal in $B(H) \otimes_h K(H)$ and in $K(H) \otimes_h B(H)$.*

Proof

By Lemma 3.1.1, with $J_1 = K(H)$, $J_2 = B(H)$ and $A = B(H)$, we have that $K(H) \otimes_h K(H)$ is a closed ideal of both $B(H) \otimes_h K(H)$ and $K(H) \otimes_h B(H)$.

Now let J be a closed ideal in $B(H) \otimes_h K(H)$ with $K(H) \otimes_h K(H) \subset J$ strictly. Let $\{e_{ij}\}$ be a set of matrix units for $K(H)$: that is, choose an orthonormal basis $\{\xi_j\}$

for H and let $e_{ij} = \xi_i \otimes \xi_j$. Define

$$J_{e_{11}} = \{x \in B(H) : x \otimes e_{11} \in J\}.$$

Then $J_{e_{11}}$ is a closed ideal in $B(H)$ for if $w, y \in B(H)$ then

$$wxy \otimes e_{11} = (w \otimes e_{11})(x \otimes e_{11})(y \otimes e_{11}) \in J.$$

Therefore $J_{e_{11}} = K(H)$ or $B(H)$.

Let $u \in J \backslash K(H) \otimes_h K(H)$, say $u = \sum_{j \in I} a_j \otimes b_j$. Using Lemma 2.1.8, decompose I so that $I = I_1 \cup I_2 \cup I_3$

where $a_j = 0$ *for* $j \in I_1$,

$a_j \in K(H)$ *for* $j \in I_2$ *and* $\{a_j : j \in I_2\}$ is strongly independent,

and $\{a_j : j \in I_3\}$ is strongly independent over $K(H)$.

Let $u' = \sum_{j \in I_3} a_j \otimes b_j$. Then $u' = u - \sum_{j \in I_2} a_j \otimes b_j \in J$ and $u' \neq 0$ since $u \notin K(H) \otimes_h K(H)$.

Consider $(1 \otimes e_{1s})u'(1 \otimes e_{t1}) \in J$,

$$\begin{aligned}
(1 \otimes e_{1s})u'(1 \otimes e_{t1}) &= \sum_{j \in I_3} a_j \otimes e_{1s} b_j e_{t1} \\
&= \sum_{j \in I_3} a_j \otimes e_{11} e_{1s} b_j e_{t1} \\
&= \left(\sum_{j \in I_3} Tr(e_{1s} b_j e_{t1}) a_j \right) \otimes e_{11}
\end{aligned}$$

We know that $u' \neq 0$ so there must exist s and t such that $(1 \otimes e_{1s})u'(1 \otimes e_{t1}) \neq 0$. But then $\sum_{j \in I_3} Tr(e_{1s} b_j e_{t1}) a_j \neq 0$. Now if $\psi : B(H) \to \mathbb{C}$ is given by

$$\psi(x) = Tr(e_{1s} x e_{t1}),$$

then ψ is a linear functional with $\|\psi\| \leqslant 1$ and

$$\sum_{j \in I_3} Tr(e_{1s} b_j e_{t1}) a_j = R_\psi(u').$$

Since $\{a_j : j \in I_3\}$ is strongly independent over $K(H)$, $R_\psi(u') \notin K(H)$. That is, $\sum_{j \in I_3} Tr(e_{1s} b_j e_{t1}) a_j \in J_{e_{11}} \backslash K(H)$. Since $R_\psi(u') \neq 0$ and $K(H)$ is the maximal ideal of $B(H)$, we must have $J_{e_{11}} = B(H)$.

Thus $B(H) \odot e_{11} \subseteq J$. Now let e be any rank one projection and let v be the partial isometry linking e and e_{11}, so that $ve_{11}v^* = e$. Then $(1 \otimes v)(x \otimes e_{11})(1 \otimes v^*) = x \otimes e$, so $B(H) \otimes e \subseteq J$ for any rank one projection e. Any finite rank projection may be written as a finite sum of rank ones so $B(H) \otimes f \subseteq J$ for any finite rank projection f.

Now let y be a finite rank operator in $B(H)$ and let e be the finite rank projection in $B(H)$ onto the range of y. Then, for any $x \in B(H)$, $x \otimes e \in J$ so

$$x \otimes y = (x \otimes e)(1 \otimes y) \in J.$$

Hence $x \otimes y \in J$ for any $x \in B(H)$ and any finite rank operator $y \in B(H)$.

Now take $z \in K(H)$. Then, given any $\varepsilon > 0$, there exists a finite rank operator y with $\|z - y\| < \varepsilon$ Then $\|x \otimes z - x \otimes y\| < \varepsilon\|x\|$ so, since J is linear and closed, $x \otimes z \in J$. Closing up in the Haagerup norm, we see that $B(H) \otimes_h K(H) \subseteq J$. That is, $B(H) \otimes_h K(H) = J$.

The proof that $K(H) \otimes_h K(H)$ is maximal in $K(H) \otimes_h B(H)$ is similar. □

Theorem *Let H be a Hilbert space. Let J be a closed ideal of $B(H) \otimes_h B(H)$ properly containing $K(H) \otimes_h K(H)$. If J does not contain $B(H) \otimes_h K(H)$, then $J = K(H) \otimes_h B(H)$ and if J does not contain $K(H) \otimes_h B(H)$, then $J = B(H) \otimes_h K(H)$.*

Proof

Without loss of generality, suppose that J does not contain $B(H) \otimes_h K(H)$. For each non-zero $\psi \in B(H)^*$ the closed linear space J_ψ defined to be the closure of

$$\left\{ \sum_j a_j \psi(b_j) : \sum_j a_j \otimes b_j \in J \right\}$$

is a closed ideal in $B(H)$. This follows because $\sum_j a_j \otimes b_j \in J$ and $a \in B(H)$ implies that $\sum_j aa_j \otimes b_j \in J$ and $\sum_j a_j a \otimes b_j \in J$. Hence the ideal is equal to $B(H)$ or $K(H)$.

Let $\{e_{ij}\}$ be a set of matrix units for $K(H)$: that is, choose an orthonormal basis $\{\xi_j\}$ for H and let $e_{ij} = \xi_i \otimes \xi_j$.

Suppose that the closed ideal $J_\psi = B(H)$ for $\psi(x) = < x\xi_t, \xi_s >$ $(x \in B(H))$ for some t and s. The ideal

$$\left\{ \sum_j a_j \psi(b_j) : \sum_j a_j \otimes b_j \in J \right\}$$

is dense in J_ψ so is also equal to $B(H)$; hence

$$1 = \sum_j a_j \psi(b_j)$$

for some $u = \sum_j a_j \otimes b_j \in J$.

Now $(1 \otimes e_{1s})u(1 \otimes e_{t1})$ is in J and equals

$$\begin{aligned}(1 \otimes e_{1s})u(1 \otimes e_{t1}) &= \sum_j a_j \otimes e_{1s} b_j e_{t1} = \sum_j a_j \otimes \psi(b_j) e_{11} \\ &= 1 \otimes e_{11}\end{aligned}$$

because $e_{1s} b e_{t1} = < b\xi_t, \xi_s > e_{11}$.

Hence, following the argument of Theorem 3.1.5, all $1 \otimes K(H)$ is contained in J, and so $B(H) \otimes_h K(H)$ is contained in J contrary to supposition.

Thus $J_\psi = K(H)$ for all functionals of the form $\psi(x) = < x\xi_t, \xi_s >$ for some $t, s \in \mathbf{N}$. Hence $J_\psi = K(H)$ by linearity in ψ for all $\psi \in K(H)^*$, because $K(H)^*$ is isomorphic to the trace class operators.

If $\psi \in B(H)^*$, then there is a net $(\psi_\lambda : \lambda \in \Lambda)$ in $K(H)^*$ such that $\|\psi_\lambda\| \leqslant \|\psi\|$ and $\psi_\lambda(b) \to \psi(b)$ for all $b \in B(H)$ by Alaoglu's theorem.

If $u = \sum_j a_j \otimes b_j \in J$, then $\sum_j a_j \psi_\lambda(b_j) \in K(H)$ for all $\lambda \in \Lambda$. Further $\sum_j a_j \psi(b_j) \to \sum_j a_j \psi(b_j)$ as λ runs over Λ. This follows because by definition of the Haagerup tensor norm attention may be restricted to only a finite sum: if $\varepsilon > 0$ there is a choice of representations $u = \sum_j a_j \otimes b_j$ and N such that

$$\left\| \sum_{j=N+1}^{\infty} a_j \otimes b_j \right\|_h < \varepsilon.$$

Then

$$\begin{aligned}\left\|\sum_j a_j\psi_\lambda(b_j) - \sum_j a_j\psi(b_j)\right\| &\leqslant (\|\psi\| + \|\psi_\lambda\|)\varepsilon + \left\|\sum_{j=1}^{N} a_j(\psi_\lambda(b_j) - \psi(b_j))\right\| \\ &< 2\|\psi\|\varepsilon + \sum_{j=1}^{N} \|a_j\| \cdot |\psi_\lambda(b_j) - \psi(b_j)| \\ &< (2\|\psi\| + 1)\varepsilon\end{aligned}$$

for λ sufficiently far along the net.

Hence $\sum_j a_j\psi(b_j) \in K(H)$ for all $\psi \in K(H)^*$. By Theorem 2.1.12 it follows that $J \subseteq K(H) \otimes_h B(H)$. Hence by Theorem 3.1.5, $J = K(H) \otimes_h B(H)$. □

Theorem 3.1.6 *The ideals $K(H) \otimes_h B(H)$ and $B(H) \otimes_h K(H)$ are maximal closed ideals of* $[B(H) \otimes_h K(H)] + [K(H) \otimes_h B(H)]$.

Proof

If J is a closed ideal of $[B(H) \otimes_h K(H)] + [K(H) \otimes_h B(H)]$ which contains $K(H) \otimes_h B(H)$ strictly, then applying the argument of Theorem 3.1.5, we can show that $B(H) \otimes_h K(H) \subseteq J$ and hence that $J = [B(H) \otimes_h K(H)] + [K(H) \otimes_h B(H)]$. The other case is similar. □

The ideas behind the proof of the following theorem were suggested by Professor R. R. Smith.

Theorem 3.1.7 *The unique maximal closed ideal of $B(H) \otimes_h B(H)$ is*

$$[B(H) \otimes_h K(H)] + [K(H) \otimes_h B(H)].$$

Proof

Suppose J is a closed ideal of $B(H) \otimes_h B(H)$ which is not contained in

$$[B(H) \otimes_h K(H)] + [K(H) \otimes_h B(H)].$$

Denote by θ the map from $B(H) \otimes_h B(H)$ onto $\frac{B(H)}{K(H)} \otimes_h \frac{B(H)}{K(H)}$. Then by the proof of Proposition 3.1.4, $Ker\ \theta = [B(H) \otimes_h K(H)] + [K(H) \otimes_h B(H)]$.

We will denote by $\|\cdot\|_*$ the C*-injective norm on $A \odot A$ given by

$$\|u\|_* = sup\|(\pi_1 \otimes \pi_2)(u)\|, \qquad u \in A \odot A,$$

where the supremum runs over all representations π_i of A (see [T] IV.4), and by $A \otimes_* A$ the completion of $A \odot A$ in this norm. Note that this norm is usually denoted $\|\cdot\|_{min}$ but is different from the min norm defined in §2.2.

Denote by ι the mapping from $\frac{B(H)}{K(H)} \otimes_h \frac{B(H)}{K(H)}$ to $\frac{B(H)}{K(H)} \otimes_* \frac{B(H)}{K(H)}$. Since $\|\cdot\|_* \leqslant \|\cdot\|_h$, $\|\iota\| \leqslant 1$ and ι is one-to-one (see [B1], §2 for details of these results) so $\iota(\theta(J)) \neq \{0\}$.

Now

$$\frac{B(H)}{K(H)} \otimes_h \frac{B(H)}{K(H)} \cong \frac{B(H) \otimes_h B(H)}{[B(H) \otimes_h K(H)] + [K(H) \otimes_h B(H)]}$$

so $\theta(J)$ is a closed ideal of $\frac{B(H)}{K(H)} \otimes_h \frac{B(H)}{K(H)}$. If $u \in \frac{B(H)}{K(H)} \otimes_h \frac{B(H)}{K(H)}$ and $v \in \frac{B(H)}{K(H)} \otimes_* \frac{B(H)}{K(H)}$ then, given $\varepsilon > 0$, there exists $v_0 \in \frac{B(H)}{K(H)} \odot \frac{B(H)}{K(H)}$ with $\|v - v_0\|_* < \varepsilon$. But $\iota(v_0) = v_0$ so $\iota(u)v_0 = \iota(uv_0)$. Then

$$\begin{aligned} \|\iota(u)v - \iota(uv_0)\|_* &= \|\iota(u)v - \iota(u)v_0\|_* \\ &< \varepsilon\|u\|, \end{aligned}$$

and similarly, $\|v\iota(u) - \iota(v_0u)\| < \varepsilon\|u\|$, so $\overline{\iota(\theta(J))}^*$ is a closed ideal of $\frac{B(H)}{K(H)} \otimes_* \frac{B(H)}{K(H)}$, where $\overline{X}^*$ denotes the closure of X in $\|\cdot\|_*$. But $\frac{B(H)}{K(H)} \otimes_* \frac{B(H)}{K(H)}$ is simple ([T], Corollary IV.4.21) so $\overline{\iota(\theta(J))}^* = \frac{B(H)}{K(H)} \otimes_* \frac{B(H)}{K(H)}$. Thus $\iota(\theta(J))$ is dense in $\|\cdot\|_*$ in $\frac{B(H)}{K(H)} \otimes_* \frac{B(H)}{K(H)}$. Hence, we can choose $u \in J$ with

$$\|\iota(\theta(u)) - \mathbf{1} \otimes \mathbf{1}\|_* < \frac{1}{5}.$$

Henceforth, for ease of notation, we will suppress the maps ι and θ.

Certainly $u \neq 0$ since it is close to the identity. Suppose that u has a representation $u = \sum_{j=1}^{\infty} a_j \otimes b_j$ with $a_j, b_j \in B(H)$ and $\sum_j a_j a_j^*$ and $\sum_j b_j^* b_j$ norm convergent. Then there exists $n \in \mathbb{N}$ with

$$\|\sum_{n+1}^{\infty} a_j \otimes b_j\|_h \leqslant \left\|\sum_{j=n+1}^{\infty} a_j a_j^*\right\|^{\frac{1}{2}} \cdot \left\|\sum_{j=n+1}^{\infty} b_j^* b_j\right\|^{\frac{1}{2}} < \frac{1}{5}.$$

Following [KR] we denote by $\mathcal{D}$ the set of all mappings $\psi : B(H) \to B(H)$ that can be defined by an equation of the form

$$\psi(x) = \sum_{k=1}^{m} \alpha_k U_k x U_k^*, \qquad x \in B(H),$$

where $\alpha_k \geqslant 0$, $\sum_{k=1}^{m} \alpha_k = 1$, and U_k are unitaries in $B(H)$. If $\psi \in \mathcal{D}$ then ψ is completely positive with

$$\|\psi\|_{cb} = \|\psi\| = \|\psi(\mathbf{1})\| = 1.$$

Now by [KR] Proposition 8.3.4 (a version of the Dixmier approximation theorem, see also [D] Theorem III.5.1 and [Sa] Theorem 2.1.16), there are complex numbers $\lambda_1, \ldots, \lambda_n$, $\mu_1, \ldots, \mu_n$ and $\psi_m \in \mathcal{D}$ with

$$\|\psi_m(a_j) - \lambda_j \mathbf{1}\| \to 0, \quad \text{and}$$

$$\|\psi_m(b_j) - \mu_j \mathbf{1}\| \to 0 \qquad \text{as } m \to \infty$$

for $j = 1, \ldots, n$. Now let $\varepsilon > 0$ be such that

$$n^{\frac{1}{2}} \varepsilon \left[\left(\sum_{j=1}^{n} |\mu_j|^2 \right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} \|a_j\|^2 \right)^{\frac{1}{2}} \right] < \frac{1}{5},$$

and choose m such that $\psi = \psi_m$ satisfies

$$\|\psi(a_j) - \lambda_j \mathbf{1}\| < \varepsilon \quad \text{and}$$

$$\|\psi(b_j) - \mu_j \mathbf{1}\| < \varepsilon$$

for $j = 1, \ldots, n$.

Then

$$\begin{aligned}
&\left\| \left(\sum_{j=1}^{n} \lambda_j \mu_j \right) \mathbf{1} \otimes \mathbf{1} - \sum_{j=1}^{n} \psi(a_j) \otimes \psi(b_j) \right\|_h \\
\leqslant \; & \left\| \sum_{j=1}^{n} (\lambda_j \mathbf{1} - \psi(a_j)) \otimes \mu_j \mathbf{1} \right\|_h + \left\| \sum_{j=1}^{n} \psi(a_j) \otimes (\mu_j \mathbf{1} - \psi(b_j)) \right\|_h \\
\leqslant \; & \left(\sum_{j=1}^{n} \|\lambda_j \mathbf{1} - \psi(a_j)\|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} |\mu_j|^2 \right)^{\frac{1}{2}} + \left\| \sum_{j=1}^{n} \psi(a_j)\psi(a_j)^* \right\|^{\frac{1}{2}} \left(\sum_{j=1}^{n} \|\mu_j \mathbf{1} - \psi(b_j)\|^2 \right)^{\frac{1}{2}} \\
\leqslant \; & n^{\frac{1}{2}} \varepsilon \left[\left(\sum_{j=1}^{n} |\mu_j|^2 \right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} \|a_j\|^2 \right)^{\frac{1}{2}} \right] \\
< \; & \frac{1}{5}.
\end{aligned}$$

Thus

$$\left\| \left(\sum_{j=1}^{n} \lambda_j \mu_j \right) 1 \otimes 1 - \sum_{j=1}^{\infty} \psi(a_j) \otimes \psi(b_j) \right\|_h \leqslant \left\| \sum_{j=n+1}^{\infty} \psi(a_j) \otimes \psi(b_j) \right\|_h + \frac{1}{5}.$$

Further

$$\begin{aligned} \left\| \sum_{j=n+1}^{\infty} \psi(a_j) \otimes \psi(b_j) \right\|_h &\leqslant \left\| \sum_{j=n+1}^{\infty} \psi(a_j)\psi(a_j)^* \right\|^{\frac{1}{2}} \left\| \sum_{j=n+1}^{\infty} \psi(b_j)^*\psi(b_j) \right\|^{\frac{1}{2}} \\ &\leqslant \left\| \sum_{j=n+1}^{\infty} a_j a_j^* \right\|^{\frac{1}{2}} \cdot \left\| \sum_{j=n+1}^{\infty} b_j^* b_j \right\|^{\frac{1}{2}} \end{aligned}$$

since $\|\psi\|_{cb} = 1$.

For $w \in B(H) \otimes_h B(H)$ denote by $d_h(w, J)$ the minimum distance in Haagerup norm from w to J, that is,

$$d_h(w, J) = inf\{\|w - v\|_h : v \in J\}.$$

Now multiplying an element of J by unitaries and taking convex combinations does not take us outside J, so

$$\sum_{j=1}^{\infty} \psi(a_j) \otimes \psi(b_j) \in J.$$

Thus

$$d_h\left(\sum_{j=1}^{n} \lambda_j \mu_j 1 \otimes 1, J \right) < \frac{2}{5}.$$

But

$$\begin{aligned} \left\| \sum_{j=1}^{n} a_j \otimes b_j - 1 \otimes 1 \right\|_* &\leqslant \left\| \sum_{j=1}^{\infty} a_j \otimes b_j - 1 \otimes 1 \right\|_* + \left\| \sum_{j=n+1}^{\infty} a_j \otimes b_j \right\|_* \\ &\leqslant \|u - 1 \otimes 1\|_* + \left\| \sum_{j=n+1}^{\infty} a_j \otimes b_j \right\|_h \\ &< \frac{2}{5}, \end{aligned}$$

using $\|\cdot\|_* \leqslant \|\cdot\|_h$.

Also, $\psi \otimes \psi$ is a norm reducing operator on the C*-algebra $B(H) \otimes_* B(H)$ (again since it consists simply of multiplying by unitaries and taking convex combinations) with the property that

$$(\psi \otimes \psi)(1 \otimes 1) = 1 \otimes 1.$$

Thus

$$\begin{aligned}
\left\|\sum_{j=1}^{n}\psi(a_j)\otimes\psi(b_j)-\mathbf{1}\otimes\mathbf{1}\right\|_* &= \left\|\psi\otimes\psi\left(\sum_{j=1}^{n}a_j\otimes b_j-\mathbf{1}\otimes\mathbf{1}\right)\right\|_* \\
&\leqslant \left\|\sum_{j=1}^{n}a_j\otimes b_j-\mathbf{1}\otimes\mathbf{1}\right\|_* \\
&< \frac{2}{5},
\end{aligned}$$

and hence

$$\begin{aligned}
\left\|\left(\sum_{j=1}^{n}\lambda_j\mu_j\right)\mathbf{1}\otimes\mathbf{1}-\mathbf{1}\otimes\mathbf{1}\right\|_* &\leqslant \left\|\left(\sum_{j=1}^{n}\lambda_j\mu_j\right)\mathbf{1}\otimes\mathbf{1}-\sum_{j=1}^{n}\psi(a_j)\otimes\psi(b_j)\right\|_* \\
&+ \left\|\sum_{j=1}^{n}\psi(a_j)\otimes\psi(b_j)-\mathbf{1}\otimes\mathbf{1}\right\|_* \\
&\leqslant \left\|\left(\sum_{j=1}^{n}\lambda_j\mu_j\right)\mathbf{1}\otimes\mathbf{1}-\sum_{j=1}^{n}\psi(a_j)\otimes\psi(b_j)\right\|_h+\frac{2}{5} \\
&< \frac{3}{5},
\end{aligned}$$

so that

$$\left|\sum_{j=1}^{n}\lambda_j\mu_j-1\right|<\frac{3}{5},$$

and certainly

$$\sum_{j=1}^{n}\lambda_j\mu_j\neq 0.$$

Finally,

$$\begin{aligned}
d_h(\mathbf{1}\otimes\mathbf{1},J) &\leqslant d_h\left(\sum_{j=1}^{n}\lambda_j\mu_j\mathbf{1}\otimes\mathbf{1},J\right)+\left\|\mathbf{1}\otimes\mathbf{1}-\sum_{j=1}^{n}\lambda_j\mu_j\mathbf{1}\otimes\mathbf{1}\right\|_h \\
&< \frac{2}{5}+\frac{3}{5}=1,
\end{aligned}$$

so the distance in Haagerup norm from the ideal to the identity is less than 1 and hence there is an invertible element in J.

An ideal which contains an invertible element clearly contains the identity and hence the whole algebra. Therefore $J=B(H)\otimes_h B(H)$. □

In proving the above result we have shown that for the simple C*-algebra $A=\frac{B(H)}{K(H)}$, the Calkin algebra, $A\otimes_h A$ is simple. We can then ask if this is true for all C*-algebras, that is, if A is simple is $A \otimes_h A$ simple? The answer is not known in general, but A. M. Sinclair has extended this proof to the case of AF C*-algebras.

We finally show that $K(H) \otimes_h K(H)$ is the minimal ideal of $B(H) \otimes_h B(H)$.

Lemma 3.1.8 *The closure of $F(H) \odot F(H)$ in the Haagerup norm is $K(H) \otimes_h K(H)$.*

Proof

It is clear that $F(H) \otimes_h F(H) \subseteq K(H) \otimes_h K(H)$ since all finite rank operators are compact.

Suppose then that $u \in K(H) \otimes_h K(H)$ and $\varepsilon > 0$. There exists $v = \sum_{j=1}^{n} a_j \otimes b_j$ such that

$$\|u - v\|_h < \varepsilon,$$

with $a_j,\ b_j \in K(H)$. Now, for $j = 1, \ldots, n$, there exist finite rank operators $c_j,\ d_j$ with

$$\begin{aligned} \|a_j - c_j\| &\leqslant \frac{\varepsilon}{\sqrt{n}}, \\ \|b_j - d_j\| &\leqslant \frac{\varepsilon}{\sqrt{n}}. \end{aligned}$$

Then

$$\begin{aligned} \left\| \sum_{j=1}^{n} a_j \otimes b_j - \sum_{j=1}^{n} c_j \otimes d_j \right\|_h &= \left\| \sum_{j=1}^{n} ((a_j - c_j) \otimes b_j + c_j \otimes (b_j - d_j)) \right\|_h \\ &\leqslant \left\| \sum_{j=1}^{n} (a_j - c_j)(a_j - c_j)^* \right\|^{\frac{1}{2}} \left\| \sum_{j=1}^{n} b_j^* b_j \right\|^{\frac{1}{2}} \\ &\quad + \left\| \sum_{j=1}^{n} c_j c_j^* \right\|^{\frac{1}{2}} \left\| \sum_{j=1}^{n} (b_j - d_j)^*(b_j - d_j) \right\|^{\frac{1}{2}} \\ &< \varepsilon \left(\left\| \sum_{j=1}^{n} b_j^* b_j \right\|^{\frac{1}{2}} + \left\| \sum_{j=1}^{n} c_j c_j^* \right\|^{\frac{1}{2}} \right). \end{aligned}$$

Hence

$$\left\| u - \sum_{j=1}^{n} c_j \otimes d_j \right\|_h < C\varepsilon,$$

for some constant C. Thus $K(H) \otimes_h K(H)$ is contained in the closure of $F(H) \odot F(H)$ in completely bounded operator norm. □

Proposition 3.1.9 *The minimal closed ideal of $B(H) \otimes_h B(H)$ is $K(H) \otimes_h K(H)$.*

Proof

Let J be a closed ideal of $B(H) \otimes_h B(H)$, and let $u \in J$. Then, using Lemma 2.1.8, we can find sequences $\{a_j\}$, $\{b_j\}$ in $B(H)$ with $u = \sum_j a_j \otimes b_j$, and $\{b_j\}$ strongly independent.

Suppose that for all rank one projections e, f in $B(H)$

$$\sum_j ea_je \otimes fb_jf = 0.$$

Fix e and f. Then

$$\sum_j (ea_je)x(fb_jf) = 0, \qquad \text{for all } x \in K(H).$$

Choose $\xi \in eH$, and $\eta \in fH$ with $\|\xi\| = \|\eta\| = 1$. Then $\xi \otimes \eta \in K(H)$, so

$$\sum_j ea_je(\xi \otimes \eta)fb_jf = 0.$$

That is,

$$e\left(\sum_j a_je\xi \otimes b_j^* f^* \eta\right) f = 0,$$

so

$$e\left(\sum_j a_j\xi \otimes b_j^* \eta\right) f = 0.$$

Then for all $\lambda,\ \nu \in H$,

$$\left\langle e\left(\sum_j a_j\xi \otimes b_j^* \eta\right) f\lambda, \nu\right\rangle = 0,$$

that is,

$$\sum_j < f\lambda, b_j^*\eta >< a_j\xi, e\nu >= 0.$$

Choosing $\lambda = \eta,\ \nu = \xi$, we have

$$\sum_j < b_j\eta, \eta >< a_j\xi, \xi >= 0,$$

or

$$\left\langle \left(\sum_j < a_j\xi, \xi > b_j\right) \eta, \eta\right\rangle = 0.$$

Now η was chosen arbitrarily in fH so

$$\left.\left(\sum_j < a_j\xi, \xi > b_j\right)\right|_{fH} = 0.$$

But f was an arbitrary rank one projection, so

$$\sum_j < a_j\xi, \xi > b_j = 0.$$

However, $(< a_j\xi, \xi >)$ is an l^2-sequence so by the strong independence of the b_j,

$$< a_j\xi, \xi >= 0 \qquad \text{for all } j.$$

Again, e was an arbitrary rank one projection so

$$< a_j\xi, \xi >= 0 \qquad \text{for all } \xi \in H,\ \|\xi\| \leqslant 1.$$

Then, using the polarisation identity,

$$< a_j\xi, \eta >= \frac{1}{4}\sum_{n=0}^{3} i^n < a_j(\xi + i^n\eta), (\xi + i^n\eta) >,$$

we have that

$$< a_j\xi, \eta >= 0 \qquad \text{for all } \xi, \eta \in H \text{ with } \|\xi\| = \|\eta\| = 1,$$

and so $a_j = 0$ for all j. But then $u = 0$.

Hence if $u = \sum_j a_j \otimes b_j \neq 0$ there exist rank one projections e and f in $B(H)$ such that

$$\sum_j ea_je \otimes fb_jf \neq 0.$$

Now

$$(e \otimes f)\left(\sum_j a_j \otimes b_j\right)(e \otimes f) = \sum_j ea_je \otimes fb_jf \neq 0,$$

so

$$J \ni \sum_j ea_je \otimes fb_jf \neq 0$$

However, for any rank one projection $e \in B(H)$,

$$eB(H)e = \mathbb{C}e,$$

so there exist $\mu_j,\ \lambda_j \in \mathbb{C}$ with $\lambda_j = ea_je,\ \mu_j = fb_jf$ and

$$\sum \lambda_j e \otimes \mu_j f \in J.$$

Now

$$\begin{aligned}
\left|\sum_j \lambda_j\mu_j\right| &\leqslant \left(\sum_j |\lambda_j|^2\right)^{\frac{1}{2}}\left(\sum_j |\mu_j|^2\right)^{\frac{1}{2}} \\
&= \left\|\sum_j (ea_je)(ea_je)^*\right\|^{\frac{1}{2}}\left\|\sum_j (fb_jf)^*(fb_jf)\right\|^{\frac{1}{2}} \\
&= \left\|e\left(\sum_j a_ja_j^*\right)e\right\|^{\frac{1}{2}}\left\|f\left(\sum_j b_j^*b_j\right)f\right\|^{\frac{1}{2}} \\
&\leqslant \left\|\sum_j a_ja_j^*\right\|^{\frac{1}{2}}\cdot\left\|\sum_j b_j^*b_j\right\|^{\frac{1}{2}} < \infty,
\end{aligned}$$

so $e \otimes f \in J$.

Now fix vectors $\xi_0, \eta_0, \lambda_0, \mu_0 \in H$ and choose vectors $\xi_1, \eta_1, \lambda_1, \mu_1 \in H$ and operators $w, z \in B(H)$ such that $e\xi_1 = \lambda_1$, $f\mu_1 = \eta_1$, $w\lambda_1 = \xi_0$,and $z^*\eta_1 = \mu_0$. Let S and T be operators on $K(H)$ given by $S = w \otimes z$ and $T = (\xi_1 \otimes \eta_0) \otimes (\lambda_0 \otimes \mu_1)$, where these operators act by multiplication on the left and right as discussed at the start of §3.1. Then if $a \in K(H)$,

$$\begin{aligned}
S(e\otimes f)T(a) &= S(e\otimes f)[(\xi_1\otimes\eta_0)a(\lambda_0\otimes\mu_1)] \\
&= S(e\otimes f)[< a\lambda_0, \eta_0 > \xi_1 \otimes \mu_1] \\
&= S < a\lambda_0, \eta_0 > e\xi_1 \otimes f\mu_1 \\
&= S < a\lambda_0, \eta_0 > \lambda_1 \otimes \eta_1 \\
&= < a\lambda_0, \eta_0 > w\lambda_1 \otimes z^*\eta_1 \\
&= < a\lambda_0, \eta_0 > \xi_0 \otimes \mu_0 \\
&= [(\xi_0\otimes\mu_0)\otimes(\lambda_0\otimes\mu_0)]a,
\end{aligned}$$

so $S(e\otimes f)T = (\xi_0 \otimes \mu_0) \otimes (\lambda_0 \otimes \mu_0)$. Now $e \otimes f \in J$, so $(\xi_0 \otimes \mu_0) \otimes (\lambda_0 \otimes \mu_0) \in J$. But ξ_0, η_0, λ_0, μ_0 were arbitrary vectors in H so if x and y are rank one operators in $B(H)$ then $x \otimes y \in J$. Then J is linear so $F(H) \odot F(H) \subseteq J$. Hence $K(H) \otimes_h K(H) \subseteq J$ by Lemma 3.1.8, so $K(H) \otimes_h K(H)$ is the minimal closed ideal of $B(H) \otimes_h B(H)$. □

Thus any ideal of $B(H) \otimes_h B(H)$ contains $K(H) \otimes_h K(H)$. By the previous results it can only be one of the ideals already identified.

We can summarise all the closed ideals of $B(H) \otimes_h B(H)$ in a diagram.

$$
\begin{tikzcd}
 & B(H) \otimes_h B(H) \arrow[d, dash] & \\
 & {[B(H) \otimes_h K(H)]+[K(H) \otimes_h B(H)]} \arrow[dl, dash] \arrow[dr, dash] & \\
B(H) \otimes_h K(H) \arrow[dr, dash] & & K(H) \otimes_h B(H) \arrow[dl, dash] \\
 & K(H) \otimes_h K(H) \arrow[d, dash] & \\
 & 0 &
\end{tikzcd}
$$

3.2 Ideals of $B(H) \otimes_{w^*h} B(H)$

We now turn our attention to the weak*-Haagerup tensor product, always using the extended definition at the end of §2.2.

Notice that when we extended the definition of the weak*-Haagerup tensor product to C*-algebras, $B(H) \otimes_{w^*h} K(H)$ was not defined as a completion of $B(H) \odot K(H)$, so it is not immediately clear that $B(H) \otimes_{w^*h} K(H)$ is a closed ideal of $B(H) \otimes_{w^*h} B(H)$.

Lemma 3.2.1 *If J is a closed ideal of a unital C*-algebra A, then $J \otimes_{w^*h} A$ is a closed ideal of $A \otimes_{w^*h} A$.*

Proof

Suppose $u \in J \otimes_{w^*h} A$ is given by

$$u = \sum_j a_j \otimes b_j, \qquad a_j \in J,\ b_j \in A,$$

where the sums $\sum_j a_j a_j^*$ and $\sum_j b_j^* b_j$ converge strongly. Suppose also that

$$\sum_i c_i \otimes d_i \text{ and } \sum_k f_k \otimes g_k \in A \otimes_{w^*h} A.$$

Then

$$\left(\sum_i c_i \otimes d_i\right)\left(\sum_j a_j \otimes b_j\right)\left(\sum_k f_k \otimes g_k\right) = \sum_{i,j,k} c_i a_j f_k \otimes d_i b_j g_k,$$

and $c_i a_j f_k \in J$, $d_i b_j g_k \in A$, and $\sum_{i,j,k} c_i a_j f_k (c_i a_j f_k)^*$ and $\sum_{i,j,k} (d_i b_j g_k)^* (d_i b_j g_k)$ converge strongly since $A \otimes_{w^*h} A$ is a Banach algebra.

The proof that $J \otimes_{w^*h} A$ is closed is similar to the proof that $A \otimes_{w^*h} B$ is a Banach space, given at the end of §2.2. □

We must now show that the ideals of the form $J_1 \otimes_{w^*h} J_2$ where J_1 and J_2 are either $K(H)$ or $B(H)$ are different.

Lemma 3.2.2 *The elementary tensor $\mathbf{1} \otimes \mathbf{1}$ is not an element of $B(H) \otimes_{w^*h} K(H)$.*

Proof

Suppose $\mathbf{1} \otimes \mathbf{1} \in B(H) \otimes_{w^*h} K(H)$. Then, given $\varepsilon > 0$, there exist sequences $\{a_j\}$ in $B(H)$, and $\{b_j\}$ in $K(H)$ with

$$\sum_j a_j x b_j = x, \qquad x \in K(H),$$

where the sum converges strongly, $\| \sum_j a_j a_j^* \| \leqslant 1 + \varepsilon$ and $\| \sum b_j^* b_j \| \leqslant 1 + \varepsilon$ and these sums converge strongly.

If $\xi,\ \eta \in H$ with $\|\xi\| = \|\eta\| = 1$ then $\xi \otimes \eta \in K(H)$ so

$$\sum_j a_j (\xi \otimes \eta) b_j = (\xi \otimes \eta),$$

and if $\lambda,\ \nu \in H$ then

$$\left\langle \left(\sum_j a_j \xi \otimes b_j^* \eta \right) \lambda, \nu \right\rangle = < \lambda, \eta >< \xi, \nu > .$$

Choosing $\lambda = \eta,\ \nu = \xi$ we have

$$\sum_j < a_j \xi, \xi >< b_j \eta, \eta > = 1.$$

Now

$$\begin{aligned} \sum_j | < a_j \xi, \xi > |^2 &= \sum_j | < \xi, a_j^* \xi > |^2 \\ &\leqslant \sum_j \|a_j^* \xi\|^2 \\ &= \sum_j < a_j^* \xi, a_j^* \xi > \\ &\leqslant \left\| \sum_j a_j a_j^* \right\| \\ &\leqslant (1 + \varepsilon)^2, \end{aligned}$$

so $(< a_j \xi, \xi >)$ is an l^2-sequence with

$$\|(< a_j \xi, \xi >)\|_2 \leqslant 1 + \varepsilon.$$

Similarly,

$$\|(< b_j \eta, \eta >)\|_2 \leqslant 1 + \varepsilon.$$

Hence, there exists N such that, for fixed ξ,

$$\sum_{j=N+1}^{\infty} | < a_j \xi, \xi > |^2 < \varepsilon.$$

Thus

$$\left| 1 - \sum_{j=1}^{N} < a_j \xi, \xi >< b_j \eta, \eta > \right| = \left| \sum_{j=N+1}^{\infty} < a_j \xi, \xi >< b_j \eta, \eta > \right|$$

$$
\begin{aligned}
&\leqslant \left(\sum_{j=N+1}^{\infty} |< a_j\xi,\xi >|^2\right)^{\frac{1}{2}} \left(\sum_{j=N+1}^{\infty} |< b_j\eta,\eta >|^2\right)^{\frac{1}{2}} \\
&< \varepsilon \left\|\sum_{j=1}^{\infty} b_j^* b_j\right\|^{\frac{1}{2}}
\end{aligned}
$$

uniformly in $\eta \in H$ with $\|\eta\| = 1$.

Now if $\{\eta_k\}_{k=1}^{\infty}$ is an orthonormal basis of H, for fixed j, there exists K_j such that

$$
|< b_j\eta_k, \eta_k >| < \frac{\varepsilon}{\sqrt{N}}, \qquad \text{if } k > K_j.
$$

Let $K = max\{K_1, \ldots, K_N\}$ and choose $k > K$. Then

$$
\begin{aligned}
\left|1 - \sum_{j=1}^{N} < a_j\xi,\xi >< b_j\eta_k,\eta_k >\right| &\geqslant 1 - \left|\sum_{j=1}^{N} < a_j\xi,\xi >< b_j\eta_k,\eta_k >\right| \\
&\geqslant 1 - \left(\sum_{j=1}^{N} |< a_j\xi,\xi >|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{N} |< b_j\eta_k,\eta_k >|^2\right)^{\frac{1}{2}} \\
&> 1 - \varepsilon \left\|\sum_j a_j a_j^*\right\|^{\frac{1}{2}},
\end{aligned}
$$

so

$$
\begin{aligned}
1 - \varepsilon(1+\varepsilon) &< 1 - \left|\sum_{j=1}^{N} < a_j\xi,\xi >< b_j\eta_k,\eta_k >\right| \\
&< \varepsilon(1+\varepsilon),
\end{aligned}
$$

which gives a contradiction, for sufficiently small ε. □

Lemma 3.2.3 *If* $0 \neq y \in K(H)$ *then* $\mathbf{1} \otimes y \notin K(H) \otimes_{w^*h} K(H)$.

Proof

Fix $0 \neq y \in K(H)$, and $\varepsilon > 0$, and suppose that $\mathbf{1} \otimes y \in K(H) \otimes_{w^*h} K(H)$. There are then sequences $\{a_j\}$, $\{b_j\}$ in $K(H)$ with

$$
\sum_j a_j x b_j = xy, \qquad x \in K(H),
$$

where the sum converges strongly, $\|\sum_j a_j a_j^*\|^{\frac{1}{2}} \leqslant 1 + \varepsilon$ and $\|\sum_j b_j^* b_j\|^{\frac{1}{2}} \leqslant 1 + \varepsilon$. As in the previous lemma, for any ξ, $\eta \in H$ with $\|\xi\| = \|\eta\| = 1$,

$$
\sum_j < a_j\xi,\xi >< b_j\eta,\eta > = < y\eta,\eta >,
$$

so there exists N which depends on η alone such that

$$\left| < y\eta, \eta > - \sum_{j=1}^{N} < a_j\xi, \xi >< b_j\eta, \eta > \right| < \varepsilon.$$

Now suppose that $\{\xi_k\}_{k=1}^{\infty}$ is an orthonormal basis of H. Then there exists K_j such that

$$| < a_j\xi_k, \xi_k > | < \frac{\varepsilon}{\sqrt{N}}, \qquad \text{if } k > K_j.$$

Choose $k > max\{K_1, \ldots, K_N\}$. Then for any $\eta \in H$ with $\|\eta\| = 1$,

$$\begin{aligned}
\varepsilon &> \left| < y\eta, \eta > - \sum_{j=1}^{N} < a_j\xi_k, \xi_k >< b_j\eta, \eta > \right| \\
&\geqslant | < y\eta, \eta > | - \left(\sum_{j=1}^{N} | < a_j\xi_k, \xi_k > |^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{N} | < b_j\eta, \eta > |^2 \right)^{\frac{1}{2}} \\
&> | < y\eta, \eta > | - \varepsilon \left\| \sum_j b_j^* b_j \right\|^{\frac{1}{2}} \\
&> | < y\eta, \eta > | - \varepsilon(1 + \varepsilon),
\end{aligned}$$

which gives a contradiction, provided ε is small enough . □

Although we did not explicitly assume the compactness of b_j in the above lemma, we can apply Lemma 2.1.8 to get the b_j strongly independent and then the compactness of y will ensure that b_j is compact.

These two lemmas, and identical arguments, show that $B(H) \otimes_{w^*h} B(H)$, $B(H) \otimes_{w^*h} K(H)$, $K(H) \otimes_{w^*h} B(H)$, and $K(H) \otimes_{w^*h} K(H)$ are all distinct. It is clear that $B(H) \otimes_{w^*h} K(H)$ and $K(H) \otimes_{w^*h} B(H)$ are distinct for if $x \in K(H)$ then $\mathbf{1} \otimes x$ is an element of the former but not the latter, and $x \otimes \mathbf{1}$ is an element of the latter but not the former.

It is clear that the set of finite rank operators on $K(H)$ is an ideal, so if we close up in the completely bounded norm we will get a closed ideal of $B(H) \otimes_{w^*h} B(H)$. As we saw in §3.1, this ideal is $K(H) \otimes_h K(H)$ and is in fact the minimal ideal of $B(H) \otimes_{w^*h} B(H)$ following the proof of its minimality in $B(H) \otimes_h B(H)$.

Lemma 3.2.4 *The closure of the completely bounded finite rank operators on $K(H)$ in the completely bounded operator norm is equal to $K(H) \otimes_h K(H)$.*

Proof

Suppose φ is a rank one operator on $K(H)$. Then we can write

$$\varphi(x) = t(x)k, \; x \in K(H)$$

for some $t \in T(H)$, $k \in K(H)$, where by $t(x)$ we mean the natural dual action of $T(H)$ on $K(H)$ given by $t(x) = Tr(tx)$. Now

$$t(x) = \sum_{j=1}^{\infty} < x\xi_j, \eta_j >,$$

for some sequences $\{\xi_j\}$, $\{\eta_j\} \in H$ with $\sum_j \|\xi_j\|^2 < \infty$, $\sum_j \|\eta_j\|^2 < \infty$ (see [Pe]), and we can write

$$k = \sum_{j=1}^{\infty} \mu_j \nu_j \otimes \zeta_j,$$

where $\{\mu_j\}$ are the eigenvalues of $(k^*k)^{\frac{1}{2}}$ and $\{\nu_j\}$ and $\{\zeta_j\}$ are orthonormal sequences in H, (see [Ri]).

Let $\tilde{\varphi} : K(H) \to K(H)$ be given by

$$\tilde{\varphi}(x) = \sum_{i,j=1}^{\infty} \mu_i(\zeta_i \otimes \eta_j)x(\xi_j \otimes \nu_i).$$

Then for $\alpha \in H$,

$$\begin{aligned} \tilde{\varphi}(x)\alpha &= \sum_{i,j=1}^{\infty} \mu_i < \alpha, \nu_i >< x\xi_j, \eta_j > \zeta_i \\ &= \varphi(x)\alpha. \end{aligned}$$

Now

$$\begin{aligned} \left\| \sum_{i,j=1}^{m,n} (\xi_j \otimes \nu_i)^*(\xi_j \otimes \nu_i) \right\| &= \left\| \sum_{i,j=1}^{m,n} (\nu_i \otimes \xi_j)(\xi_j \otimes \nu_i) \right\| \\ &= sup \left\{ \left| \sum_{i,j=1}^{m,n} < \alpha, \nu_i >< \xi_j, \xi_j >< \nu_i, \alpha > \right| : \alpha \in H, \; \|\alpha\| \leqslant 1 \right\} \\ &= \left(\sum_{j=1}^{n} \|\xi_j\|^2 \right) sup \left\{ \sum_{i=1}^{m} | < \alpha, \nu_i > |^2 : \alpha \in H, \; \|\alpha\| \leqslant 1 \right\} \\ &\leqslant \left(\sum_{j=1}^{\infty} \|\xi_j\|^2 \right) < \infty, \end{aligned}$$

so $\sum_{i,j}(\xi_j \otimes \nu_i)^*(\xi_j \otimes \nu_j)$ converges in norm. Similarly,

$$\left\| \sum_{i,j=1}^{m,n} \mu_i(\zeta_i \otimes \eta_j)[\mu_i(\zeta_i \otimes \eta_j)]^* \right\| = sup \left\{ \sum_{i,j=1}^{m,n} |\mu_i|^2 | < \alpha, \zeta_i > |^2 \|\eta_j\|^2 : \alpha \in H, \ \|\alpha\| \leqslant 1 \right\}$$
$$\leqslant \|k\|^2 \sum_{j=1}^{n} \|\eta_j\|^2 < \infty.$$

Hence $\tilde{\varphi} \in K(H) \otimes_h K(H)$. Thus $K(H) \otimes_h K(H)$ contains all the rank one operators on $K(H)$ and so contains all finite rank operators on $K(H)$ by linearity. Since $F(K(H))$ is clearly an ideal in $B(K(H))$, the closure of the finite ranks in the completely bounded norm must be a closed ideal of $B(H) \otimes_{w^*h} B(H)$. Since we have shown that $K(H) \otimes_h K(H)$ is minimal, they must be equal. □

Now $K(K(H))$ is clearly an ideal in $B(K(H))$ so the completely bounded norm closure of the completely bounded compact operators on $K(H)$ is a closed ideal in $B(H) \otimes_{w^*h} B(H)$, again using the isometric isomorphism with $CB(K(H), B(H))$. The question is whether it is different from $K(H) \otimes_h K(H)$.

Proposition 3.2.5 *Let $\varphi : K(H) \to B(H)$ be a completely bounded compact operator. Then there are sequences $\{a_j\}$ and $\{b_j\}$ in $K(H)$ such that*

(i) $\varphi(x) = \sum_j a_j x b_j, \qquad x \in K(H),$

(ii) $\|\varphi\|_{cb} = \left\| \sum_j a_j a_j^* \right\|^{\frac{1}{2}} \cdot \left\| \sum_j b_j^* b_j \right\|^{\frac{1}{2}}$*, with the sums converging strongly,*

and $\{a_j\}$, $\{b_j\}$ are strongly independent.

We will need to use a lemma for the proof.

Lemma 3.2.6 *Let $\varphi : K(H) \to B(H)$ be a compact operator and for fixed ξ, η in H define $T : H \to H$ by*

$$< T\mu, \nu > = < \varphi(\xi \otimes \nu)\mu, \eta >, \qquad \mu, \ \eta \in H.$$

Then T is compact with $\|T\| \leqslant \|\varphi\| \cdot \|\xi\| \cdot \|\eta\|$.

Proof

Clearly T is continuous with $\|T\| \leqslant \|\varphi\| \cdot \|\xi\| \cdot \|\eta\|$.

Let $\mathbb{E}_n : K(H) \to K(H)$ be a conditional expectation from $K(H)$ onto a matrix subalgebra for all n, so that

$$\|\mathbb{E}_n(x) - x\| \to 0 \text{ as } n \to \infty \text{ for all } x \in K(H).$$

Since φ is compact,

$$\|\mathbb{E}_n \circ \varphi - \varphi\| \to 0 \text{ as } n \to \infty.$$

Let $T_n : H \to H$ be defined by

$$< T_n\mu, \eta > = < \mathbb{E}_n\varphi(\xi \otimes \nu)\mu, \eta >, \qquad \mu,\ \eta \in H.$$

Then T_n is a finite rank operator with

$$rank(T_n) \leqslant rank(\mathbb{E}_n),$$

and

$$\|T - T_n\| \leqslant \|\varphi - \mathbb{E}_n\varphi\| \cdot \|\xi\| \cdot \|\eta\| \to 0 \text{ as } n \to \infty.$$

Hence T is compact. □

Proof of Proposition 3.2.5

Suppose $\varphi(x) = \sum_j a_j x b_j$ with $\|\varphi\|_{cb} = \left\|\sum_j a_j a_j^*\right\|^{\frac{1}{2}} \cdot \left\|\sum_j b_j^* b_j\right\|^{\frac{1}{2}}$. We apply Lemma 2.1.8 with $W = K(H)$. Then there is a disjoint partition $J_1 \cup J_2 \cup J_3$ of $\mathbb{N}$ and a unitary $u \in B(l^2)$ such that if $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)^t$ and if $c = ub$ and $d = au^*$ then

(i) $c_j = 0$ for $j \in J_1$,

(ii) $c_j \in K(H)$ for $j \in J_2$, and $\{c_j : j \in J_2\}$ is strongly independent,

(iii) $\{c_j : j \in J_3\}$ is strongly independent over $K(H)$,

where c_j are the components of c.

Now

$$\begin{aligned}
\varphi(x) &= a(x \otimes 1)b \\
&= a(1 \otimes u^*)(x \otimes 1)(1 \otimes u)b \\
&= d(x \otimes 1)c.
\end{aligned}$$

Fix $\xi,\ \eta \in H$. If $T : H \to H$ is defined for $\mu,\ \nu \in H$ by

$$\begin{aligned}
< T\mu, \nu > &= < \varphi(\xi \otimes \nu)\mu, \eta > \\
&= \sum_j < d_j(\xi \otimes \nu)c_j\mu, \eta > \\
&= \sum_j < c_j\mu, \nu >< d_j\xi, \eta >,
\end{aligned}$$

then T is compact by the preceding lemma.

If $\psi : B(H) \to \mathbb{C}$ is given by

$$\psi(x) =< x\xi, \eta >, \qquad x \in B(H),$$

then $\|\psi\| \leqslant \|\xi\| \cdot \|\eta\|$, so $\psi \in B(H)^*$, and $R_\psi(u) = \sum_j \psi(d_j)c_j$ converges in norm for all $u \in B(H) \otimes_{w^*h} B(H)$. Thus

$$T - \sum_{j \in J_2} < d_j\xi, \eta > c_j = \sum_{j \in J_3} < d_j\xi, \eta > c_j \in K(H).$$

But $(< d_j\xi, \eta >) \in l^2$ since $d \in B(H^\infty, H)$, so by the strong independence of $c_j,\ j \in J_3$, we have

$$< d_j\xi, \eta >= 0, \qquad \text{for all } j \in J_3,$$

and, since ξ and η were arbitrarily chosen, hence $d_j = 0$ for $j \in J_3$, so we can renumber J_2 and get

$$\varphi(x) = \sum_j c_j x d_j, \qquad x \in K(H),$$

with $c_j \in K(H)$ strongly independent and

$$\|\varphi\|_{cb} = \left\|\sum_j c_j c_j^*\right\|^{\frac{1}{2}} \cdot \left\|\sum_j d_j^* d_j\right\|^{\frac{1}{2}}.$$

Repeating the process with φ^*, where

$$\varphi^*(x) = \sum_j d_j^* x c_j^*,$$

will give $d_j \in K(H)$ strongly independent. □

The above proposition shows that the completely bounded norm closure of the compact completely bounded operators on $K(H)$ is contained in $K(H) \otimes_{w^*h} K(H)$. The inclusion is strict, as can be seen from the following lemma.

Lemma 3.2.7 *There is an element u of $K(H) \otimes_{w^*h} K(H)$ such that φ_u is not compact.*

Proof

Let $\{e_{ij}\}$ be the matrix units of $B(H)$, and let $u = \sum_j e_{jj} \otimes e_{jj}$.

Now $\{e_{jj}\}$ is a bounded sequence in $B(H)$, and

$$\varphi_u(e_{jj}) = e_{jj}, \qquad \text{for all } j.$$

Thus φ_u maps a bounded sequence to a bounded but not convergent sequence, so φ_u is not compact. □

We have shown that a compact operator on $K(H)$ may be written

$$\varphi(x) = \sum_j a_j x b_j,$$

with $a_j, b_j \in K(H)$, and strong convergence of the appropriate sums but we need norm convergence if we are to write $\varphi = V^*\pi(x)W$ with V and W compact.

Proposition 3.2.8 *There is a compact completely bounded operator on $K(H)$ which is not approximable by finite rank operators in the completely bounded norm.*

Proof

Let

$$C_0 = \bigoplus_{n+1}^{\infty} M_n(\mathbb{C})$$

be the c_0-direct sum of $M_n(\mathbb{C})$, so that if $x \in C_0$ with $x = (x_1, x_2, \ldots)$, $x_n \in M_n(\mathbb{C})$, then $\|x_n\| \to 0$ as $n \to \infty$ and $\|x\| = sup\{\|x_n\| : n \in \mathbb{N}\}$. Let $I : C_0 \to K(H)$ be an embedding of C_0 in $K(H)$: if $\{\xi_j\}_{j=1}^{\infty}$ is a suitable orthonormal basis of H, and $\xi \in H$ with $\xi = \sum_j \lambda_j \xi_j$, then

$$(Ix)\xi = x_1\lambda_1\xi_1 + x_2 \begin{pmatrix} \lambda_2\xi_2 \\ \lambda_3\xi_3 \end{pmatrix} + x_3 \begin{pmatrix} \lambda_4\xi_4 \\ \lambda_5\xi_5 \\ \lambda_6\xi_6 \end{pmatrix} + \ldots$$

The compactness of Ix is guaranteed by $\|x_n\| \to 0$ as $n \to \infty$, since if $\varepsilon > 0$ there exists N such that if $n > N$ then $\|x_n\| < \varepsilon$. Now let $y_N \in C_0$ be defined by $y_N = (x_1, \ldots, x_N, 0, \ldots)$: then Iy_N is certainly finite rank and $\|Ix - Iy_N\| < \varepsilon$; hence Ix is compact.

Denote by A_n the image of the nth component of C_0 in $K(H)$ and let $\mathsf{E}_n : K(H) \to A_n$ be a conditional expectation onto A_n. Let $\{e_{ij}^{(n)}\}_{1 \leqslant i,j \leqslant n}$ be the matrix units of A_n, and let $T^{(n)}$ be the transpose map on $M_n(\mathbb{C})$.

Then $\|T^{(n)}\| = 1$ and if $T_j^{(n)} : M_j(M_n(\mathbb{C})) \to M_j(M_n(\mathbb{C}))$ is the amplification $T_j^{(n)} = T^{(n)} \otimes \iota_j$ then

$$\|T_j^{(n)}\| = \begin{cases} j & 1 \leqslant j \leqslant n \\ n & j > n, \end{cases}$$

so $\|T^{(n)}\|_{cb} = n$ (see Theorem 2.1.1).

Define $\varphi : K(H) \to K(H)$ by

$$\varphi(x) = \bigoplus_{n=1}^{\infty} \frac{1}{n} T^{(n)} \mathsf{E}_n(x), \qquad x \in K(H).$$

Then

$$\|\varphi\| = sup\{\|\mathsf{E}_n \varphi\| : n \in \mathbb{N}\},$$

and by permuting rows and columns of $j \times j$ matrices we get that

$$\|\varphi_j\| = sup\{\|(\mathsf{E}_n \varphi)_j\| : n \in \mathbb{N}\},$$

so

$$\begin{aligned} \|\varphi\|_{cb} &= sup\{\|\mathsf{E}_n \varphi\|_{cb} : n \in \mathbb{N}\} \\ &= sup\left\{\left\|\frac{1}{n} T^{(n)} \mathsf{E}_n\right\|_{cb} : n \in \mathbb{N}\right\} \\ &= 1. \end{aligned}$$

Further,

$$\begin{aligned} \left\|\varphi - \bigoplus_{n=1}^{k} \left(\frac{1}{n} T^{(n)} \mathsf{E}_n\right)\right\| &= \left\|\bigoplus_{n=k+1}^{\infty} \left(\frac{1}{n} T^{(n)} \mathsf{E}_n\right)\right\| \\ &= sup\left\{\left\|\frac{1}{n} T^{(n)} \mathsf{E}_n\right\| : n \geqslant k+1\right\} \\ &= \frac{1}{k+1} \to 0 \qquad \text{as } k \to \infty. \end{aligned}$$

But $\frac{1}{n}T^{(n)}\mathbb{E}_n$ is a finite rank operator on $K(H)$ for all n, so φ is compact.

Suppose now that ψ is a finite rank operator on $K(H)$ with

$$\|\psi\|_{cb} \leqslant 1 \text{ and } \|\varphi - \psi\|_{cb} < \frac{1}{2}.$$

Then

$$\|\mathbb{E}_n\varphi - \mathbb{E}_n\psi\|_{cb} \leqslant \frac{1}{2} \qquad \text{for all } n.$$

Now for $x \in K(H)$,

$$\mathbb{E}_n(x) = \sum_{i,j=1}^{n} Tr(xe_{ij}^{(n)})e_{ji}^{(n)},$$

where Tr is the trace on $T(H)$ with $Tr(e_{11}) = 1$. Suppose that ψ is rank one, so $\psi(x) = Tr(xb)y$ for some fixed $y \in K(H)$ and $b \in T(H)$. For any linear functional f, $\|f\|_{cb} = \|f\|$ (Proposition 1.2.6) so

$$\|\mathbb{E}_n\psi\|_{cb} \leqslant |Tr(b)| \cdot \|\mathbb{E}_n y\|_{cb} \to 0 \qquad \text{as } n \to \infty,$$

since $y \in K(H)$. But any finite rank operator is a linear combination of rank one operators, so we have

$$\|\mathbb{E}_n\psi\|_{cb} \to 0 \qquad \text{as } n \to \infty.$$

But

$$\begin{aligned} \|\mathbb{E}_n\varphi - \mathbb{E}_n\psi\|_{cb} &\geqslant \big| \|\mathbb{E}_n\varphi\|_{cb} - \|\mathbb{E}_n\psi\|_{cb}\big| \\ &\to 1 \qquad \text{as } n \to \infty, \end{aligned}$$

which contradicts the assumption that

$$\|\mathbb{E}_n\varphi - \mathbb{E}_n\psi\|_{cb} \leqslant \frac{1}{2}.$$

Hence φ cannot be approximated in completely bounded norm by finite rank operators. □

This example shows that we cannot write a general compact completely bounded operator on $K(H)$ as $V^*\pi(x)W$, choosing V and W compact.

Of course, the operator constructed above is approximable in operator norm by finite ranks (indeed that is how we showed it was compact), which it must be since

$K(H)$ has the approximation property. As we noted before, $B(H)$ does not have the approximation property so not all compact operators on $B(H)$ are approximable in operator norm by finite rank operators. It would be interesting to know whether there is an example of a completely bounded compact !operator on $B(H)$ which is not approximable by finite ranks in operator norm. For example is the operator constructed by Szankowski [Sz] completely bounded?

Now the set of compact completely bounded operators on $K(H)$, which we denote by $CB \cap K(K(H))$ and the closure of the finite rank operators on $K(H)$ are clearly closed ideals of $B(H) \otimes_{w^*h} B(H)$. Thus we can now summarise the ideals found in this section in a diagram. We have shown that they are different but they are by no means exhaustive.

$$
\begin{tikzcd}
 & B(H) \otimes_{w^*h} B(H) \arrow[dl, dash] \arrow[dr, dash] & \\
B(H) \otimes_{w^*h} K(H) \arrow[dr, dash] & & K(H) \otimes_{w^*h} B(H) \arrow[dl, dash] \\
 & K(H) \otimes_{w^*h} K(H) \arrow[d, dash] & \\
 & CB \cap K(K(H)) \arrow[d, dash] & \\
 & K(H) \otimes_h K(H) \arrow[d, dash] & \\
 & 0 &
\end{tikzcd}
$$

4 Completely bounded p-summing operators

In this chapter, we ask a similar question to that asked in the last chapter: if φ is a completely bounded operator and is p-summing for $1 \leqslant p < \infty$, can we say anything about the representation of φ?

The first section gives some general background on p-summing operators, in particular Pietsch's theorem. The second section details how far we can answer the given question. For more background on p-summing operators see the books of Pisier [Pi4] and Jameson [J].

We find a necessary condition for $p \geqslant 2$ and a sufficient condition for $1 \leqslant p \leqslant 2$, summarising the case $p = 2$ in a corollary at the end of the chapter.

4.1 p-summing operators and Pietsch's theorem

The theory of p-summing operators (or 'p-absolutely summing' operators) is, in large part, due to Pietsch, although some of the ideas go back to Grothendieck (Pietsch's '1-absolutely summing' was 'pré-intégral à droite' in Grothendieck's Résumé [G2]).

Definition 4.1.1 *Let X, Y be Banach spaces and $\varphi : X \to Y$, and let $0 < p < \infty$. Then φ is p-summing if there is a constant C such that for all finite subsets $\{x_j\}$ in X, we have*

$$\left(\sum_{j=1}^{n} \|\varphi(x_j)\|^p\right)^{\frac{1}{p}} \leqslant C \ sup \left\{ \left(\sum_{j=1}^{n} |f(x_j)|^p\right)^{\frac{1}{p}} : f \in X^*, \ \|f\| \leqslant 1 \right\}.$$

We denote by $\pi_p(\varphi)$ the smallest constant C satisfying this inequality, and by $\Pi_p(X,Y)$ the set of all p-summing operators $\varphi : X \to Y$.

Lemma 4.1.2 *If $1 \leqslant p < \infty$ then π_p is a norm on $\Pi_p(X,Y)$, with respect to which the space is complete.*

Proof

Suppose $1 \leqslant p < \infty$ and φ, $\psi \in \Pi_p(X,Y)$. Then if $x_1, \ldots, x_n \in X$, by Minkowski's inequality,

$$\left(\sum_{j=1}^{n}\|\varphi(x_j)+\psi(x_j)\|^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{j=1}^{n}\|\varphi(x_j)\|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n}\|\psi(x_j)\|^p\right)^{\frac{1}{p}}$$
$$\leqslant (\pi_p(\varphi)+\pi_p(\psi))\, sup\left\{\left(\sum_{j=1}^{n}|f(x_j)|^p\right)^{\frac{1}{p}} : f \in X^*,\ \|f\| \leqslant 1\right\},$$

so $\pi_p(\varphi+\psi) \leqslant \pi_p(\varphi)+\pi_p(\psi)$.

Clearly, if $z \in \mathbb{C}$ then $\pi_p(z\varphi) = |z|\pi_p(\varphi)$, and if $\pi_p(\varphi) = 0$ then $\|\varphi(x)\| = 0$ for all $x \in X$ and so $\varphi \equiv 0$.

It follows from the definition that $\Pi_p(X,Y)$ is complete with respect to π_p. □

If $0 < p < 1$, Minkowski gives the reverse inequality so $\Pi_p(X,Y)$ is not a Banach space, but we will not concern ourselves with this case.

The Pietsch factorisation theorem is the most important result concerning p-summing operators and we will have much cause to use it in various guises in this and subsequent chapters.

Theorem 4.1.3 *Let $\varphi : X \to Y$ be a p-summing operator and let K be the unit ball of X^* equipped with the weak*-topology.*

(i) Let $C = \pi_p(\varphi)$. Then there is a probability measure λ on K such that

$$\|\varphi(x)\| \leqslant C\left(\int_K |f(x)|^p d\lambda(f)\right)^{\frac{1}{p}}.$$

(ii) Conversely, any operator satisfying this inequality for some constant C is p-summing and $\pi_p(\varphi) \leqslant C$.

The proof is an application of the Hahn-Banach theorem, the measure arising because we consider continuous functions on K and the measure is the linear functional given by the Hahn-Banach theorem. The theorem can be stated in terms of finding a bound with an element of the dual, see chapter 5 for example.

Corollary 4.1.4 *If $0 < p < q < \infty$, then $\Pi_p(X,Y) \subset \Pi_q(X,Y)$ and $\pi_q(\varphi) \leqslant \pi_p(\varphi)$ for all $\varphi : X \to Y$.*

The proof is an application of Pietsch's theorem, noting that the L_p norm over a probability space is a non-decreasing function of p.

The next corollary makes explicit that Pietsch's theorem is in fact a factorisation theorem.

Corollary 4.1.5 *Let $\varphi : X \to Y$ be a p-summing operator. Then φ factors as $\varphi = \psi \circ \iota \circ \theta$, where*

$$X \xrightarrow{\theta} C(K) \xrightarrow{\iota} L_p(K,\lambda) \xrightarrow{\psi} Y,$$

where K is a compact set, λ is a probability measure on K, ι is the natural inclusion map and ψ which is defined on the closure of $(\iota\circ\theta)(X)$ *satisfies $\|\psi\| \leqslant \pi_p(\varphi)$.*

This corollary is immediate from the proof of Pietsch's theorem, as $L_p(K,\lambda)$ is given in the theorem.

It will be useful to note the following result from Lindenstrauss and Tzafriri's book.

Theorem 4.1.6 ([LT], Volume I Theorem 2.b.4) *If H is a Hilbert space then for every $1 \leqslant p < \infty$ the space $\Pi_p(H,H)$ consists exactly of the Hilbert-Schmidt operators on H, $HS(H)$.*

The proof uses the classical inequality of Khinchin.

4.2 Completely bounded p-summing operators

Given a C*-algebra A and a Hilbert space H and a completely bounded operator $\varphi : A \to B(H)$ which has a representation

$$\varphi(x) = V^* \rho(x) W, \qquad x \in A,$$

where ρ is a representation of A on some Hilbert space K, and $V,\ W : H \to K$, then we would like to find necessary and sufficient conditions on V and W for φ to be p-summing. In this section we consider the extent to which this is possible.

For notational simplicity, we will follow Jameson's notation for the quantity occurring on the right hand side of the inequality defining p-summing.

Definition 4.2.1 *Let $p \geqslant 1$ and let $\{x_j\}$ be a finite sequence of elements of a Banach space X. Then we define*

$$\mu_p(x_1, \ldots, x_n) = sup \left\{ \left(\sum_{j=1}^{n} |f(x_j)|^p \right)^{\frac{1}{p}} : f \in X^*,\ \|f\| \leqslant 1 \right\}.$$

We will need to work with this quantity a great deal so the following lemma of Jameson's which relates it to something which is easier to calculate will be very useful.

Lemma 4.2.2 ([J]) *Let $1 < p < \infty$. If $x_1, \ldots, x_n$ is a finite sequence in a normed space X then*

$$\mu_p(x_1, \ldots, x_n) = sup \left\{ \left\| \sum_{j=1}^{n} \alpha_j x_j \right\| : \sum_{j=1}^{n} |\alpha_j|^{p'} \leqslant 1 \right\},$$

where $\frac{1}{p'} + \frac{1}{p} = 1$.

We will need to calculate this quantity for rank one elements of $B(H)$.

Lemma 4.2.3 *If $\{\xi_j\}$ is an orthonormal basis in a Hilbert space H, then for $n \in \mathbb{N}$,*

$$\mu_p(\{\xi_j \otimes \xi_k\}_{j,k=1}^{n}) = \begin{cases} n^{\frac{2}{p}-1} & 1 \leqslant p < 2 \\ 1 & p \geqslant 2. \end{cases} \tag{9}$$

Proof

By the previous lemma,

$$\mu_p(\{\xi_j \otimes \xi_k\}_{j,k=1}^{n}) = sup \left\{ \left\| \sum_{j,k=1}^{n} \alpha_{jk} \xi_j \otimes \xi_k \right\| : \sum_{j,k=1}^{n} |\alpha_{jk}|^{p'} \leqslant 1 \right\}. \tag{10}$$

Now

$$\left\|\sum_{j,k=1}^{n} \alpha_{jk}\xi_j \otimes \xi_k\right\| = sup\left\{\left\|\sum_{j,k=1}^{n} \alpha_{jk} < \lambda, \xi_k > \xi_j\right\| : \lambda \in H,\ \|\lambda\| \leqslant 1\right\}$$

$$= sup\left\{\left(\sum_{j=1}^{n}\left|\sum_{k=1}^{n} \alpha_{jk} < \lambda, \xi_k >\right|^2\right)^{\frac{1}{2}} : \lambda \in H,\ \|\lambda\| \leqslant 1\right\}$$

$$\leqslant sup\left\{\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}|\alpha_{jk}|^2\right)\left(\sum_{k=1}^{n}| < \lambda, \xi_k > |^2\right)\right)^{\frac{1}{2}} : \lambda \in H,\ \|\lambda\| \leqslant 1\right\}$$

$$\leqslant \left(\sum_{j,k=1}^{n}|\alpha_{jk}|^2\right)^{\frac{1}{2}}.$$

But $(l^p, \|\cdot\|_p) \subseteq (l^q, \|\cdot\|_q)$ for $p \leqslant q$ so

$$\left(\sum_{j,k=1}^{n}|\alpha_{jk}|^2\right)^{\frac{1}{2}} \leqslant \left(\sum_{j,k=1}^{n}|\alpha_{jk}|^{p'}\right)^{\frac{1}{p'}},$$

for $p' \leqslant 2$; that is for $p \geqslant 2$.

Thus if $p \geqslant 2$ then $\mu_p(\{\xi_j \otimes \xi_k\}_{j,k=1}^{n}) \leqslant 1$.

Now putting

$$\alpha_{ij} = \begin{cases} 1 & i = j = 1 \\ 0 & \text{otherwise} \end{cases}$$

in (2) we get $\mu_p(\{\xi_j \otimes \xi_k\}_{j,k=1}^{n}) \geqslant 1$ for $p \geqslant 1$. Hence $\mu_p(\{\xi_j \otimes \xi_k\}_{j,k=1}^{n}) = 1$ for $p \geqslant 2$.

By Hölder's inequality, if $p < 2$ and $\frac{1}{2} = \frac{1}{p'} + \frac{1}{q}$ then

$$\left(\sum_{j,k=1}^{n}|\alpha_{jk}|^2\right)^{\frac{1}{2}} \leqslant \left(\sum_{j,k=1}^{n}|\alpha_{jk}|^{p'}\right)^{\frac{1}{p'}}\left(\sum_{j,k=1}^{n}1^q\right)^{\frac{1}{q}}.$$

But

$$\left(\sum_{j,k=1}^{n}1^q\right)^{\frac{1}{q}} = n^{\frac{2}{q}} = n^{\frac{2}{p}-1},$$

so if $p < 2$

$$\mu_p(\{\xi_j \otimes \xi_k\}_{j,k=1}^{n}) \leqslant n^{\frac{2}{p}-1}.$$

However, putting $\alpha_{jk} = n^{\frac{-2}{p'}} = n^{\frac{2}{p}-2}$ we have

$$\sum_{j,k=1}^{n}|\alpha_{jk}|^{p'} = n^2.n^{-2} = 1,$$

and, since

$$\left\|\sum_{j,k=1}^{n}\xi_j \otimes \xi_k\right\| = n,$$

$$\mu_p(\{\xi_j \otimes \xi_k\}_{j,k=1}^n) = n^{\frac{2}{p}-1} \text{ for } 1 \leqslant p < 2$$ □

We can now prove a result about the representation of a completely bounded 2-summing map.

Theorem 4.2.4 *Let $\varphi : B(H) \to B(K)$ be completely bounded and suppose φ has a representation given by*

$$\varphi(x) = VxW^*, \qquad x \in B(H),$$

where $V,\ W : H \to K$. If φ is 2-summing then $(V^*V)^{\frac{1}{2}}$, $(W^*W)^{\frac{1}{2}} \in C_2(H)$ and $\|V\|_2 \cdot \|W\|_2 \leqslant \pi_2(\varphi)$.

Proof

Suppose $\{\xi_s\}$ is an orthonormal basis of H.

Then $V(\xi_s \otimes \xi_t)W^* = V\xi_s \otimes W\xi_t$ so

$$\|V(\xi_s \otimes \xi_t)W^*\| = \|V\xi_s\| \cdot \|W\xi_t\|.$$

Thus

$$\begin{aligned} \sum_{s,t} \|V(\xi_s \otimes \xi_t)W^*\|^2 &= \left(\sum_s \|V\xi_s\|^2\right)\left(\sum_t \|W\xi_t\|^2\right) \\ &= \|V\|_2^2 \|W\|_2^2. \end{aligned}$$

Now φ is 2-summing, so for any $n \in \mathbb{N}$,

$$\begin{aligned} \sum_{s,t=1}^n \|V(\xi_s \otimes \xi_t)W^*\|^2 &\leqslant \pi_2(\varphi)^2 \mu_2(\{\xi_s \otimes \xi_t\}_{s,t=1}^n)^2 \\ &\leqslant \pi_2(\varphi)^2, \end{aligned}$$

by the previous lemma.

Therefore

$$\|V\|_2^2\|W\|_2^2 = \sum_{s,t} \|V(\xi_s \otimes \xi_t)W^*\|^2 \leqslant \pi_2(\varphi)^2,$$

so the result follows. □

The problem in proving this theorem for a general C*-algebra is that mentioned in the introduction, of not knowing how the C*-algebra sits inside $B(H)$ which makes calculations involving basis vectors of H difficult.

From Lemma 4.2.2 and Theorem 4.2.4 we can see that if $1 \leqslant p < 2$ and $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is given by $\varphi(x) = VxW^*$ and φ is completely bounded then

$$\|V\|_p^p \|W\|_p^p \leqslant \pi_p(\varphi) n^{\frac{2}{p}-1}.$$

This is actually best possible as can be seen by considering $p = 1$.

Lemma 4.2.5 ([J] 2.2) *Let $x_1, \ldots, x_n$ be elements of a normed linear space X. Then*

$$\mu_1(x_1, \ldots, x_n) = sup\left\{ \left\| \sum_{j=1}^{n} \alpha_j x_j \right\| : |\alpha_j| \leqslant 1 \; j = 1, \ldots, n \right\}.$$

This is really just Lemma 4.2.2 for $p = 1$.

Lemma 4.2.6 ([J] 3.3) *If φ is 1-summing then it is 2-summing and $\pi_2(\varphi) \leqslant \pi_1(\varphi)$.*

Lemma 4.2.7 ([J] 5.13) *Let X be any n-dimensional normed linear space. Then $\pi_2(I_X) = \sqrt{n}$.*

Let $\{e_{ij}\}$ be the matrix units of $M_n(\mathbb{C})$. Then

$$\sum_{j=1}^{n} \|e_{jj}\| = n,$$

but by Lemma 4.2.5 $\mu_1(e_{11}, \ldots, e_{nn}) = 1$. Thus if I is the identity on $M_n(\mathbb{C})$, then $\pi_1(I) \leqslant n$.

However, by Lemmas 4.2.6 and 4.2.7 , $\pi_1(I) \geqslant \pi_2(I) = n$. Hence $\pi_1(I) = n$.

Now, writing $I(x) = IxI$, $x \in M_n(\mathbb{C})$, we have $\|I\|_1 = n$, so our inequality becomes

$$\|I\|_1 \|I\|_1 = n^2 \leqslant \pi_1(I) n = n^2.$$

We now prove a result in the opposite direction.

Theorem 4.2.8 *Suppose A is C^*-algebra and $\varphi : A \to B(H)$ is a completely bounded operator with representation*

$$\varphi(x) = V\rho(x)W^*, \qquad x \in A,$$

where ρ is a representation of A on a Hilbert space K and $V,\ W \in B(K,H)$.

If $1 \leqslant p \leqslant 2$ and $V, W \in C_p(K,H)$ then $\varphi : A \to C_p(H)$ and is p-summing in the p-norm.

Proof

For all p, $C_p(K,H)$ is an ideal in $B(K,H)$ so $\varphi(x) \in C_p(H)$ for all $x \in A$.

Suppose that $V = \sum s_j \xi_j \otimes \eta_j$ and $W = \sum t_j \lambda_j \otimes \mu_j$, where $\{\mu_j\}, \{\eta_j\}, \{\xi_j\},\ \{\lambda_j\}$ are orthonormal sequences in K and H, and $s_j,\ t_j \in \mathbb{C}$.

Then

$$\begin{aligned} V\rho(x)W^* &= \sum_{j,k} s_j \overline{t_k} (\xi_j \otimes \eta_j)\rho(x)(\mu_k \otimes \lambda_k) \\ &= \sum_k \overline{t_k} \left(\sum_j s_j < \rho(x)\mu_k, \eta_j > \xi_j \right) \otimes \lambda_k, \end{aligned}$$

so

$$\begin{aligned} \|V\rho(x)W^*\|_p^p &\leqslant \sum_k |t_k|^p \left\| \sum_j s_j < \rho(x)\mu_k, \eta_j > \xi_j \right\|^p \\ &= \sum_k |t_k|^p \left(\sum_j |s_j|^2 | < \rho(x)\mu_k, \eta_j > |^2 \right)^{\frac{p}{2}}. \end{aligned}$$

But

$$\left(\sum_j |s_j|^2 | < \rho(x)\mu_k, \eta_j > |^2 \right)^{\frac{1}{2}} \leqslant \left(\sum_j |s_j|^p | < \rho(x)\mu_k, \eta_j > |^p \right)^{\frac{1}{p}},$$

for $1 \leqslant p \leqslant 2$, since $l^p \subseteq l^2$ for $1 \leqslant p \leqslant 2$ so

$$\|V\rho(x)W^*\|_p^p \leqslant \sum_{j,k} |s_j|^p |t_k|^p | < \rho(x)\mu_k, \eta_j > |^p.$$

Let $\Omega = \mathit{Ball}\ K \times \mathit{Ball}\ H$. Define a probability measure on Ω by

$$m = \|V\|_p^{-p} \|W\|_p^{-p} \sum_{j,k} |s_j|^p |t_k|^p \delta(\mu_k, \eta_j),$$

where $\delta(\mu, \eta)$ is a point mass at (μ, η). Then we have

$$\int_\Omega | < \rho(x)\mu, \eta > |^p dm(\mu, \eta) = \|V\|_p^{-p} \|W\|_p^{-p} \sum_{j,k} |s_j|^p |t_k|^p | < \rho(x)\mu_k, \eta_j > |^p,$$

so

$$\|V\rho(x)W^*\|_p \leqslant \|V\|_p \|W\|_p \left(\int_\Omega | < \rho(x)\mu, \eta > |^p dm(\mu, \eta) \right)^{\frac{1}{p}}.$$

Then, by Pietsch's theorem, φ is p-summing with

$$\pi_p(\varphi) \leqslant \|V\|_p \|W\|_p.$$

□

Picking out the results for $p = 2$ from the theorems, we have the following corollary.

Corollary 4.2.9 *If $\varphi : B(H) \to B(H)$ is completely bounded and has representation*

$$\varphi(x) = VxW^*, \qquad x \in K(H),$$

where $V, W \in B(H)$, then φ is 2-summing if and only if we can choose V and W in $HS(H)$.

Proof

If φ is 2-summing then $V, W \in HS(H)$ by Theorem 4.2.4.

Conversely suppose that $V, W \in HS(H)$. Then by Theorem 4.2.8, $\varphi : (B(H), \|\cdot\|) \to (B(H), \|\cdot\|_2)$ is 2-summing, where $\|\cdot\|$ is the operator norm on $B(H)$. But $\|\cdot\| \leqslant \|\cdot\|_2$ so certainly $\varphi : (B(H), \|\cdot\|) \to (B(H), \|\cdot\|)$ is 2-summing.□

5 The Grothendieck-Pisier-Haagerup inequality

In this chapter we turn our attention to the famous inequality of Grothendieck which he called the 'fundamental theorem of the metric theory of tensor products' in his Résumé [G2].

In the first section, we briefly discuss the background of Grothendieck's work and the results of Pisier and Haagerup in extending it to a C*-algebra setting, using a symmetrised version of the modulus. Pisier's book [Pi4] contains more on these ideas in great depth. In the second section, we explore when the usual (non-symmetric) modulus can be used, particularly with the added assumption that our operators are completely bounded.

We define 2-concavity, a condition on the image space which is sufficient for all completely bounded maps to satisfy an inequality of the required type. We then show that if all completely bounded operators into a given space satisfy the inequality, then the space must be 2-concave.

5.1 Inequalities with the symmetrised modulus

The classical inequality of Grothendieck may be stated in many forms but one of the simplest is as follows.

Theorem 5.1.1 *Let X_1, X_2 be compact sets and let F be a scalar-valued bounded bilinear form on $C(X_1) \times C(X_2)$, then there are probability measures λ_1, λ_2 on X_1, X_2 respectively such that*

$$|F(x_1, x_2)| \leqslant K\|F\| \left(\int |x_1|^2 d\lambda_1 \cdot \int |x_2|^2 d\lambda_2 \right)^{\frac{1}{2}}, \qquad x_1 \in X_1,\ x_2 \in X_2,$$

where K is an absolute constant.

Grothendieck conjectured ([G2], Problème 4) that the result could be extended to general C*-algebras, although he underestimated the difficulty of the problem, thinking it a finite-dimensional problem. Throughout this section, instead of the usual modulus given by $|x| = (x^*x)^{\frac{1}{2}}$ we will use the symmetrised version given by

$$|x| = \left(\frac{x^*x + xx^*}{2} \right)^{\frac{1}{2}},$$

denoting it with bold lines to emphasise the difference.

Theorem 5.1.2 *Let A and B be two C^*-algebras. Let $V : A \times B \to \mathbb{C}$ be a bounded bilinear form. Then there exist two states f, g on A and B respectively such that*

$$|V(x,y)| \leqslant K\|V\| \left\{f(|x|^2)\right\}^{\frac{1}{2}} \left\{g(|y|^2)\right\}^{\frac{1}{2}}, \qquad x \in A,\ y \in B,$$

where K is an absolute constant.

If A and B are commutative this reduces to Theorem 5.1.1.

This result, with the added assumption that one of the C*-algebras had the bounded approximation property, was proved by Pisier [Pi1] and he later amended his proof so that only the approximation property needed to be assumed [Pi2]. Haagerup finally removed the assumption of the approximation property altogether [H3]. Kaijser and Sinclair [KS] removed the approximation property assumption from Pisier's proof and in his book [Pi4], Pisier improves his proof to remove this condition also.

Although Pisier and Haagerup state the theorem in different forms, their proofs have features in common: notably obtaining a bound in terms of states acting on fourth powers, in Haagerup's case on the imaginary part of the bilinear form

$$|Im\ V(x,y)| \leqslant 2\varphi(x^4)^{\frac{1}{4}}\psi(y^4)^{\frac{1}{4}},$$

and then improving this to a bound involving squares of elements

$$|Im\ V(x,y)| \leqslant 2\varphi'(x^2)^{\frac{1}{2}}\psi'(y^2)^{\frac{1}{2}}.$$

Haagerup initially proves the result with an additional factor of $\frac{5}{2}$, and then uses an interpolation argument to improve this estimate.

The following theorem of Effros and Kishimoto (which was first proved in an unpublished manuscript of Haagerup's [H4]) identifies $CB(A \times B, \mathbb{C})$ with the dual of $A \otimes_h B$ and shows that Haagerup's result may be considered as a statement about the dual of $C(X_1) \otimes_h C(X_2)$.

Theorem 5.1.3 ([EK], Theorem 2.1) *Suppose that A and B are unital C^*-algebras and that $\varphi : A \times B \to \mathbb{C}$ is bilinear. Then the following are equivalent.*

(i) For all finite sequences $x_1, \ldots, x_n$ in A and $y_1, \ldots, y_n$ in B,

$$\left|\sum_{j=1}^{n} \varphi(x_j, y_j)\right| \leqslant \left\|\sum_{j=1}^{n} x_j x_j^*\right\|^{\frac{1}{2}} \cdot \left\|\sum_{j=1}^{n} y_j^* y_j\right\|^{\frac{1}{2}}$$

(ii) *There exist states $f \in A^*$, $g \in B^*$ such that*

$$|\varphi(x,y)| \leqslant f(xx^*)^{\frac{1}{2}} g(y^*y)^{\frac{1}{2}}, \qquad x \in A,\ y \in B.$$

The first condition says that φ, considered as an element of the dual of $A \otimes_h B$, has norm less than or equal to 1.

Since it will be more useful to us to use the theorem in the version proved by Pisier we will state it in that form, but first we need some definitions.

Definition 5.1.4 *The Rademacher functions $\{r_j : j \in \mathbb{N}\}$, $r_j : [0,1] \to \{+1,-1\}$ are defined by*

$$r_j(t) = sign(sin(2^j\pi t)), \qquad 0 \leqslant t \leqslant 1,$$

where

$$sign(t) = \begin{cases} 1 & \textit{if } t \geqslant 0, \\ -1 & \textit{if } t < 0. \end{cases}$$

It follows from this definition that if dt is Lebesgue measure on $[0,1]$ then

$$\int_{[0,1]} r_j(t) r_k(t) dt = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 5.1.5 (i) *An operator φ between Banach spaces X and Y is said to be of type p, $1 \leqslant p \leqslant 2$ if there is a constant C such that, for all finite sequences $x_1, \ldots, x_n$ of X,*

$$\left(\int \left\| \sum_{j=1}^{n} r_j(t) \varphi(x_j) \right\|^2 dt \right)^{\frac{1}{2}} \leqslant C \left(\sum_{j=1}^{n} \|x_j\|^p \right)^{1/p}.$$

The smallest such constant C is denoted $T_p(\varphi)$.

(ii) *An operator φ between Banach spaces X and Y is said to be of cotype q, $2 \leqslant q \leqslant \infty$, if there is a constant C such that, for all finite sequences $x_1, \ldots, x_n$ of X,*

$$\left(\sum_{j=1}^{n} \|\varphi(x_j)\|^q \right)^{1/q} \leqslant C \left(\int \left\| \sum_{j=1}^{n} r_j(t) x_j \right\|^2 dt \right)^{\frac{1}{2}}.$$

The smallest such constant is denoted $C_q(\varphi)$.

(iii) *A space X is said to be of type p if the identity operator on X is of type p and of cotype q if the identity operator on X is of cotype q. We will write $T_p(X)$ for $T_p(I_X)$ and $C_q(X)$ for $C_q(I_X)$ if no confusion will arise.*

See chapter 3 of Pisier's book [Pi4] for more details on type and cotype. The most interesting cases are type 2 and cotype 2 and we shall only really be concerned with cotype 2 spaces. Some authors used to say that X has a subquadratic average rather then X is of type 2 and has a superquadratic average rather than is of cotype 2 ([TJ] for example).

Examples

1. Consider a finite subset $(h_1, \dots, h_n)$ in a Hilbert space H. Then

$$\sum \|h_j\|^2 = \int \sum_{j,k} r_j(t) r_k(t) < h_j, h_k > dt = \int \left\| \sum r_j(t) h_j \right\|^2,$$

so H is of type 2 and cotype 2, with $T_2(H) = C_2(H) = 1$.

2. The dual of a C*-algebra A is of cotype 2, with $C_2(A^*) \leqslant 2\sqrt{e}$ (see [TJ] and [Pi4]).

There is a converse to the first example, due to Kwapień [Kw].

Theorem 5.1.6 *A Banach space X is of type 2 and cotype 2 if and only if it is isomorphic to a Hilbert space H.*

The ideas behind the Grothendieck inequality and 2-summing operators are closely related to the notion of factorising an operator through a Hilbert space.

Definition 5.1.7 *Let X, Y be Banach spaces. We say that $\varphi : X \to Y$ factors through a Hilbert space if there is a Hilbert space H and operators $\psi : X \to H$ and $\theta : H \to Y$ such that $\varphi = \theta \circ \psi$.*

$$\begin{tikzcd} & H \arrow[dr, "\theta"] & \\ X \arrow[ur, "\psi"] \arrow[rr, "\varphi"] & & Y \end{tikzcd}$$

Let $\gamma_2(\varphi) = inf\{\|\psi\| \cdot \|\theta\|\}$ where the infimum runs over all possible factorisations.

The following corollary of Pietsch's theorem (see chapter 4) shows that all 2-summing operators factor through a Hilbert space.

Corollary 5.1.8 ([Pi4], Corollary 1.8) *Let $\varphi : X \to Y$ be a 2-summing operator. Then φ factors as $\varphi = \psi \circ \iota \circ \theta$*

$$X \xrightarrow{\psi} C(K) \xrightarrow{\iota} L_2(K, \lambda) \xrightarrow{\theta} Y,$$

where K is a compact set, λ is a probability measure on K, ι is the natural inclusion map and θ is such that $\|\theta\| \leqslant \pi_2(\varphi)$.

Conversely, any operator of this form is 2-summing with $\pi_2(\varphi) \leqslant \|\theta\|$.

Proposition 5.1.9 ([Pi4], Theorem 9.8) *Let $\varphi : A \to Y$ be an operator from a C^*-algebra A into a Banach space Y and let C be a constant. Then the following are equivalent.*

(i) There is a state f on A such that

$$\|\varphi(x)\| \leqslant C \left(f(|x|^2)\right)^{\frac{1}{2}}, \qquad x \in A.$$

(ii) For all finite sequences $x_1, \ldots, x_n$ in A we have

$$\sum_{j=1}^{n} \|\varphi(x_j)\|^2 \leqslant C^2 \left\| \sum_{j=1}^{n} |x_j|^2 \right\|.$$

When (i) and (ii) are satisfied we will say that φ is a 2-C^*-summing operator.

Proof

Suppose *(i)* holds. Then

$$\begin{aligned}
\sum_{j=1}^{n} \|\varphi(x_j)\|^2 &\leqslant C^2 \sum_{j=1}^{n} f(|x_j|^2) \\
&= C^2 f\left(\sum_{j=1}^{n} |x_j|^2\right) \\
&\leqslant C^2 \|f\| \left\| \sum_{j=1}^{n} |x_j|^2 \right\| \\
&= C^2 \left\| \sum_{j=1}^{n} |x_j|^2 \right\|.
\end{aligned}$$

Conversely, suppose that *(ii)* holds. Recall that if f is a linear functional on A, then f is positive, denoted $f \geqslant 0$ if and only if $f(x^*x) \geqslant 0$ for all $x \in A$.

Let K be the set of states on A; that is

$$K = \{f \in A^* : f \geqslant 0, \ \|f\| \leqslant 1\}.$$

Then ([Pe] §3.2) K is convex and weak*-compact.

Let $\mathcal{C}$ be the collection of continuous functions on K of the form

$$F_{(x_1,\dots,x_n)}(f) = C^2 \sum_{j=1}^{n} f(|x_j|^2) - \sum_{j=1}^{n} \|\varphi(x_j)\|^2.$$

Then $\mathcal{C}$ is a convex cone in $C(K)$ since for $0 \leqslant \lambda, \mu \leqslant 1$ with $\lambda + \mu = 1$ and $x_1, \dots, x_m, x_{m+1}, \dots, x_n$ in A,

$$\lambda F_{(x_1,\dots,x_m)} + \mu F_{(x_{m+1},\dots,x_n)} = F_{(\sqrt{\lambda}x_1,\dots,\sqrt{\lambda}x_m,\sqrt{\mu}x_{m+1},\dots,\sqrt{\mu}x_n)}.$$

Now if $x \in A$, $\|x\| = sup\{|f(x)| : f \in K\}$, so for any sequence $x_1, \dots, x_n$ in A

$$\begin{aligned} sup\left\{\left|F_{(x_1,\dots,x_n)}(f)\right| : f \in K\right\} &= C(\varphi)^2 \left\|\sum_{j=1}^{n} |x_j|^2\right\| - \sum_{j=1}^{n} \|\varphi(x_j)\|^2 \\ &\geqslant 0 \end{aligned}$$

by condition *(ii)*. Hence $\mathcal{C}$ is disjoint from the open cone $\mathcal{O} = \{\psi \in C(K) : max\ \psi < 0\}$. Notice that $\mathcal{C}$ and $\mathcal{O}$ are both sets of real-valued functions. Then the Hahn-Banach theorem gives a real-valued functional g on the real-valued continuous functions on K which is negative on $\mathcal{O}$ and non-negative on $\mathcal{C}$. We can then complexify g by defining $g(x + iy) = g(x) + ig(y)$ and the complex-valued functional will have the same norm as the real-valued functional, so that g is defined on all of $C(K)$. Since the Riesz representation theorem says that the dual of $C(K)$ is the set of Radon measures on K, we can identify g with a positive measure λ satisfying

$$\lambda(\psi) \begin{cases} \geqslant 0 & \text{for } \psi \in \mathcal{C} \\ < 0 & \text{for } \psi \in \mathcal{O}. \end{cases}$$

Since K is weak*-compact, we can replace λ by $\lambda(K)^{-1}\lambda$ to get a probability measure on K.

Then let $f \in A^*$ be given by

$$f(x) = \int_K k(x) d\lambda(k), \qquad x \in A,$$

and condition *(i)* follows. □

Pisier calls the following result the 'little Grothendieck Theorem'.

Theorem 5.1.10 ([Pi4], Theorem 9.4) *Let $\varphi : A \to H$ be an operator from a C*-algebra A into a Hilbert space H. Then there is a state f on A such that*

$$\|\varphi(x)\| \leqslant 2\|\varphi\| \left\{ f(|x|^2) \right\}^{\frac{1}{2}}, \qquad x \in A.$$

Thus this theorem says that any operator from a C*-algebra into a Hilbert space is 2-C*-summing.

To prove his version of the inequality in a C*-algebra setting, Pisier applies this result to an abstract version of the Grothendieck Theorem.

Theorem 5.1.11 ([Pi4], Theorem 4.1) *Let X, Y be Banach spaces such that X^* and Y are of cotype 2. Then every approximable operator $\varphi : X \to Y$ factors through a Hilbert space, with $\gamma_2(\varphi) \leqslant (2C_2(X^*)C_2(Y))^{\frac{3}{2}} \|\varphi\|$.*

Note that in this abstract version of the Grothendieck inequality, the approximability assumption cannot in general be removed. This can be seen with an example of Pisier which first appeared in [Pi3] and is reproduced in his book.

Theorem 5.1.12 ([Pi4], Theorem 10.6) *Any Banach space of cotype 2 can be embedded isometrically into a Banach space X of cotype 2 such that*

(i) $X \check{\otimes} X = X \hat{\otimes} X$.

(ii) X and X^* are both of cotype 2, and $B(X, l^2) = \Pi_1(X, l^2)$, $B(X^*, l^2) = \Pi_1(X^*, l^2)$.

(iii) The natural map $X^* \hat{\otimes} X \to X^* \check{\otimes} X$ is surjective.

If we could apply Theorem 5.1.11 without the approximability assumption to the identity operator on this space X then it would tell us that X is isomorphic to a Hilbert space. But the identity on l^2 is not 1-summing, so $B(l^2) \neq \Pi_1(l^2)$ and hence we cannot have $B(X, l^2) = \Pi_1(X, l^2)$.

Putting together Theorems 5.1.10 and 5.1.11 we get the desired inequality.

Theorem 5.1.13 *Let A be a C^*-algebra and let Y be a Banach space of cotype 2. Then any approximable operator $\varphi : A \to Y$ factors through a Hilbert space. Moreover there is a state f on A such that*

$$\|\varphi(x)\| \leqslant C\|\varphi\| \left\{f(|x|^2)\right\}^{\frac{1}{2}}, \qquad x \in A$$

where C is a constant depending only on the cotype 2 constant of Y.

Proof

By Theorem 5.1.11 there is a factorisation of φ given by $\varphi = \theta \circ \psi$ where $\psi : A \to H$ and $\theta : H \to Y$. Then by Theorem 5.1.10 there is a state f on A such that

$$\|\psi(x)\| \leqslant 2\|\psi\| \left\{f(|x|^2)\right\}^{\frac{1}{2}}, \qquad x \in A.$$

Hence

$$\begin{aligned} \|\varphi(x)\| &\leqslant \|\theta\| \cdot \|\psi(x)\| \\ &\leqslant 2\|\theta\| \cdot \|\psi\| \left\{f(|x|^2)\right\}^{\frac{1}{2}}, \end{aligned}$$

for $x \in A$, since f is a state. Now construct a net of factorisations of $\varphi = \theta_n \circ \psi_n$ with associated states f_n such that

$$\|\varphi(x)\| \leqslant 2\|\theta_n\| \cdot \|\psi_n\| \left\{f_n(|x|^2)\right\}^{\frac{1}{2}}, \qquad x \in A,$$

and $\|\theta_n\| \cdot \|\psi_n\| \to \gamma_2(\varphi)$. The set of states on A is weak*-compact so there is a subnet of $\{f_n\}$ which converges weak* to a state f. Thus

$$\begin{aligned} \|\varphi(x)\| &\leqslant 2\gamma_2(\varphi) \left\{f(|x|^2)\right\}^{\frac{1}{2}} \\ &\leqslant 2(C_2(A^*)C_2(Y))^{\frac{3}{2}} \|\varphi\| \left\{f(|x|^2)\right\}^{\frac{1}{2}} \\ &\leqslant 2(2\sqrt{e}C_2(Y))^{\frac{3}{2}} \|\varphi\| \left\{f(|x|^2)\right\}^{\frac{1}{2}}, \end{aligned}$$

as required. □

Theorem 5.1.10 can be proved directly (as it is in [Pi4]) or follows easily from Theorem 5.1.2 if we neglect the value of the constant: given $\varphi : A \to H$ define

$V : A \times A \to \mathbb{C}$ by $V(x,y) = < \varphi(x^*), \varphi(y) >$ and notice that $\|\varphi(x)\|^2 = V(x^*, x)$, so $\|V\| = \|\varphi\|^2$. Then

$$|V(x^*,x)| \leqslant K\|V\| \left\{f(|x|^2)\right\}^{\frac{1}{2}} \left\{g(|x|^2)\right\}^{\frac{1}{2}}$$

becomes

$$\|\varphi(x)\| \leqslant K'\|\varphi\| \left\{f'(|x|^2)\right\}^{\frac{1}{2}},$$

where $f' = \left(\frac{f+g}{2}\right)$ and K' is a new constant.

A special case of Theorem 5.1.13 can similarly be derived from Theorem 5.1.2. If Y is taken to be the dual of a C*-algebra B, then define $V : A \times B \to \mathbb{C}$ by $V(x,y) = \varphi(x)y$, and the desired inequality follows.

5.2 Inequalities with the usual modulus

In this section we replace the symmetrised modulus of the previous section with the usual C*-algebra modulus $|x| = (x^*x)^{\frac{1}{2}}$, and try to see when we can prove similar inequalities.

Theorem 5.1.2 is false in general if the usual modulus is used, as can be seen from the following example of Haagerup [H3]. Suppose there exists a constant C such that

$$|V(x,y)| \leqslant C\|V\| f\,(x^*x)^{\frac{1}{2}}\, g\,(yy^*)^{\frac{1}{2}}\,.$$

Take $A = B = B(H)$, the bounded operators on an infinite-dimensional Hilbert space, and the bilinear form $V(x,y) = \omega(xy)$ where ω is a fixed state on $B(H)$. For any $n \in \mathbb{N}$ we can choose isometries $u_1, \ldots, u_n \in B(H)$ such that $u_k u_k^*$, $k = 1, \ldots, n$ are orthogonal projections with sum 1. Then

$$\begin{aligned} |\sum_{k=1}^{n} V(u_k^*, u_k)| &\leqslant C\|V\|\, f\left(\sum_{j=1}^{n} u_k u_k^*\right)^{\frac{1}{2}} g\left(\sum_{j=1}^{n} u_k u_k^*\right)^{\frac{1}{2}} \\ &\leqslant C\|V\|. \end{aligned}$$

However, $\|V\| = 1$ and $\sum V(u_k^*, u_k) = \sum \omega(u_k^* u_k) = n$ which is a contradiction.

We can define a notion of 2-C*-summing with the usual modulus exactly as with the symmetrised modulus and then obtain an analogy of Theorem 5.1.9.

Proposition 5.2.1 *Let $\varphi : A \to B$ is a map between C*-algebras A and B and let C be a constant. Then the following are equivalent.*

(i) There is a state f on A such that

$$\|\varphi(x)\| \leqslant C f\,(x^*x)^{\frac{1}{2}}\,, \qquad ,x \in A.$$

(ii) For all finite sequences $x_1, \ldots, x_n \in A$ we have

$$\left(\sum_{j=1}^{n} \|\varphi(x_j)\|^2\right)^{\frac{1}{2}} \leqslant C \left\|\sum_{j=1}^{n} x_j^* x_j\right\|^{\frac{1}{2}}.$$

The proof is identical to the version with the symmetrised modulus proved in the previous section, but with the symmetrised modulus replaced by $|x| = (x^*x)^{\frac{1}{2}}$.

As we have seen that the most general form of Grothendieck's theorem does not hold if we use the usual modulus, we consider whether or not it will hold if we restrict our attention to completely bounded maps. In fact for a bilinear completely bounded map, it is easily seen that Grothendieck's inequality follows from the definition, which was one of the motivations for defining completely bounded operators in this way.

Let $\varphi : A \times B \to \mathbb{C}$ be a completely bounded bilinear map from the direct product of two C*-algebras into $\mathbb{C}$ (see §1.2 for the definition). Then

$$\left\| \varphi_n \left(\begin{pmatrix} x_1 \dots x_n \\ 0 \dots 0 \end{pmatrix}, \begin{pmatrix} y_1 & 0 \\ \vdots & \vdots \\ y_n & 0 \end{pmatrix} \right) \right\| \leqslant \|\varphi\|_{cb} \left\| \begin{pmatrix} x_1 \dots x_n \\ 0 \dots 0 \end{pmatrix} \right\| \left\| \begin{pmatrix} y_1 & 0 \\ \vdots & \vdots \\ y_n & 0 \end{pmatrix} \right\|.$$

for $x_1, \dots, x_n \in A, y_1, \dots, y_n \in B$, so

$$\left\| \begin{pmatrix} \sum \varphi(x_j, y_j) & 0 \\ 0 & 0 \end{pmatrix} \right\| \leqslant \|\varphi\|_{cb} \left\| \begin{pmatrix} x_1 \dots x_n \\ 0 \dots 0 \end{pmatrix} \right\| \left\| \begin{pmatrix} y_1 & 0 \\ \vdots & \vdots \\ y_n & 0 \end{pmatrix} \right\|,$$

and hence

$$\left| \sum_{j=1}^{n} \varphi(x_j, y_j) \right| \leqslant \|\varphi\|_{cb} \left\| \sum_{j=1}^{n} x_j x_j^* \right\|^{\frac{1}{2}} \left\| \sum_{j=1}^{n} y_j^* y_j \right\|^{\frac{1}{2}}.$$

We can now apply the result of Effros and Kishimoto, stated in the previous section as Theorem 5.1.3.

Thus for our completely bounded bilinear map there exist states f on A and g on B such that

$$|\varphi(x, y)| \leqslant \|\varphi\|_{cb} f(xx^*)^{\frac{1}{2}} g(y^*y)^{\frac{1}{2}}, \qquad x \in A,\ y \in B.$$

Recall the matrix norm structure for a dual space defined in §2.2, namely that

$$\|(f_{ij})\| = sup\{\|(f_{ij})(x_{kl})\| : \|(x_{kl})\| \leqslant 1\}.$$

Then the following result easily follows.

Corollary 5.2.2 *Let A and B be C^*-algebras and let $\varphi : A \to B^*$ be completely bounded. Then there exists a state f on A such that*

$$\|\varphi(x)\| \leqslant \|\varphi\|_{cb} f(xx^*)^{\frac{1}{2}}, \qquad x \in A.$$

Proof

Define $\psi : A \times B \to \mathbb{C}$ by $\psi(x,y) = \varphi(x)y$. Then

$$\begin{aligned} \|\psi_n((x_{ij}),(y_{ij}))\| &= \left\| \left(\sum_{k=1}^{n} \psi(x_{ik}, y_{kj}) \right) \right\| \\ &= \left\| \left(\sum_{k=1}^{n} \varphi(x_{ik}) y_{kj} \right) \right\| \\ &= \|\varphi_n((x_{ij}))(y_{ij})\| . \end{aligned}$$

Hence $\|\psi\|_{cb} = \|\varphi\|_{cb}$ and so by the above, there exist states f on A and g on B such that

$$|\psi(x,y)| \leqslant \|\psi\|_{cb} f(xx^*)^{\frac{1}{2}} g(y^*y)^{\frac{1}{2}}.$$

Thus

$$|\varphi(x)y| \leqslant \|\varphi\|_{cb} f(xx^*)^{\frac{1}{2}} g(y^*y)^{\frac{1}{2}},$$

and so

$$\begin{aligned} \|\varphi(x)\| &= sup\{|\varphi(x)y| : y \in B, \|y\| = 1\} \\ &\leqslant \|\varphi\|_{cb} f(xx^*)^{\frac{1}{2}} \sup\{g(y^*y)^{\frac{1}{2}} : y \in B, \|y\| = 1\} \\ &= \|\varphi\|_{cb} f(xx^*)^{\frac{1}{2}}. \end{aligned}$$

□

The notion of 2-concavity will be useful for our purposes. The definition is the analogue of 2-concavity in a Banach lattice [LT] and is used in the C*-algebra context by Lust-Piquard and Pisier [LPP].

Definition 5.2.3 *(i) Let X, Y be subspaces of C^*-algebras. An operator $\varphi : X \to Y$ is said to be 2-concave if there exists a constant C such that for all finite sequences $x_1, \ldots, x_n$ in X*

$$\left(\sum_{j=1}^{n} \|\varphi(x_j)\|^2 \right)^{\frac{1}{2}} \leqslant C \left\| \sum_{j=1}^{n} x_j^* x_j \right\|^{\frac{1}{2}}.$$

(ii) A subspace X of a C^*-algebra A is said to be 2-concave if the identity operator on X is 2-concave.

Now, if $\{r_j\}_{j=1}^{\infty}$ are the Rademacher functions defined in §5.1, then

$$\begin{aligned}\left\| \int \left| \sum_{j=1}^{n} r_j(t) x_j \right|^2 dt \right\| &= \left\| \int \left(\sum_{j=1}^{n} r_j(t) x_j \right)^* \cdot \left(\sum_{k=1}^{n} r_k(t) x_k \right) dt \right\| \\ &= \left\| \int \left(\sum_{j,k=1}^{n} r_j(t) r_k(t) x_j^* x_k \right) dt \right\| \\ &= \left\| \sum_{j=1}^{n} x_j^* x_j \right\|,\end{aligned}$$

so we can reformulate the inequality defining 2-concavity of a subspace of a C*-algebra as

$$\left(\sum_{j=1}^{n} \|x_j\|^2 \right)^{\frac{1}{2}} \leqslant C \left\| \int \left| \sum_{j=1}^{n} r_j(t) x_j \right|^2 dt \right\|^{\frac{1}{2}}.$$

Maurey and Pisier [MP] prove Khinchin-type inequalities, namely that there exist constants K_{pq} such that

$$\left(\int \left\| \sum_{j=1}^{n} r_j(t) x_j \right\|^p dt \right)^{\frac{1}{p}} \leqslant K_{pq} \left(\int \left\| \sum_{j=1}^{n} r_j(t) x_j \right\|^q dt \right)^{\frac{1}{q}}$$

for all $1 \leqslant p, q < \infty$ using a result of Kahane [Ka].

It is interesting to note that there do not exist corresponding constants for the quantities appearing in the 2-concavity definition. In particular there does not exist a constant K such that

$$\left\| \int \left| \sum_{j=1}^{n} r_j(t) x_j \right|^4 dt \right\|^{\frac{1}{4}} \leqslant K \left\| \int \left| \sum_{j=1}^{n} r_j(t) x_j \right|^2 dt \right\|^{\frac{1}{2}}.$$

Suppose such a constant K exists, and that E is a Banach space of cotype 2.

Since $\varphi : A \to E$ is 4-C*-summing; that is there is a constant $\pi_4(\varphi)$ such that for all sequences $x_1, \ldots, x_n$ in A

$$\left(\sum_{j=1}^{n} \|\varphi(x_j)\|^4 \right)^{\frac{1}{4}} \leqslant \pi_4(\varphi) \left\| \int \left| \sum_{j=1}^{n} r_j(t) x_j \right|^4 dt \right\|^{\frac{1}{4}}.$$

Then

$$\left(\int \left\| \sum_{j=1}^{n} r_j(t) \varphi(x_j) \right\|^4 dt \right)^{\frac{1}{4}} \leqslant K \pi_4(\varphi) \left\| \int \left| \sum_{j=1}^{n} r_j(t) x_j \right|^2 dt \right\|^{\frac{1}{2}},$$

but E is of cotype 2, so

$$\left\| \int \left| \sum_{j=1}^{n} r_j(t)\varphi(x_j) \right|^4 dt \right\|^{\frac{1}{4}} \geqslant \left\| \int \left| \sum_{j=1}^{n} r_j(t)\varphi(x_j) \right|^2 dt \right\|^{\frac{1}{2}}$$

$$\geqslant \frac{1}{C_2(E)} \left(\sum_{j=1}^{n} \|\varphi(x_j)\|^2 \right)^{\frac{1}{2}},$$

and hence

$$\left(\sum_{j=1}^{n} \|\varphi(x_j)\|^2 \right)^{\frac{1}{2}} \leqslant C_2(E) K \pi_4(\varphi) \left\| \sum_{j=1}^{n} x_j^* x_j \right\|^{\frac{1}{2}}.$$

We can now use a version of the Riesz-Thorin interpolation theorem (see [BL]), as used by Jameson.

Theorem 5.2.4 ([J], Theorem 11.1) *If* $1 \leqslant p,\ q \leqslant \infty$ *and* $0 < \theta < 1$ *where* $\frac{1}{p} = \frac{\theta}{q}$ *then for any operator* φ

$$\pi_p(\varphi) \leqslant \pi_q(\varphi)^{\theta} \|\varphi\|^{1-\theta}.$$

Applying this result with $p = 4,\ q = 2$, and $\theta = \frac{1}{2}$, we have that

$$\pi_4(\varphi) \leqslant \pi_2(\varphi)^{\frac{1}{2}} \|\varphi\|^{\frac{1}{2}}.$$

If φ is finite rank, we can show that $\pi_2(\varphi) < \infty$ ([Pil], Main Theorem). Hence if φ is finite rank then

$$\left(\sum_{j=1}^{n} \|\varphi(x_j)\|^2 \right)^{\frac{1}{2}} \leqslant C_2(E) K \pi_2(\varphi)^{\frac{1}{2}} \|\varphi\|^{\frac{1}{2}} \left\| \sum_{j=1}^{n} x_j^* x_j \right\|^{\frac{1}{2}}.$$

As we have seen, this is equivalent to Theorem 5.1.13 but we have throughout used the usual (non-symmetric) modulus, and as we saw at the beginning of this section, Grothendieck's inequality is false if we use the usual modulus.

Example

A Hilbert space in a von Neumann algebra is 2-concave.

A norm-closed linear subspace H of a von Neumann algebra M is said to be a Hilbert space in M if $a \in H$ implies $a^*a \in \mathbb{C}1$ and $x \in M$ and $xa = 0$ for all $a \in H$ implies $x = 0$ (see for example [Ro]).

The space H is actually a Hilbert space: the inner product on H is given by $< b, a > 1 = b^*a$. Thus

$$\begin{aligned} \left\|\sum x_j^* x_j\right\| &= \left\|\sum < x_j, x_j > 1\right\| \\ &= \sum \|x_j\|^2 \end{aligned}$$

so H is 2-concave.

Proposition 5.2.5 *If $\varphi : A \to X$ is a completely bounded map from a C^*-algebra A into a 2-concave subspace X of a C^*-algebra B, then there exists a constant C and a state f on A such that*

$$\|\varphi(a)\| \leqslant C\|\varphi\|_{cb} f(a^*a)^{\frac{1}{2}}, \qquad a \in A.$$

Proof

It is sufficient by Proposition 5.2.1 to show that for all finite sequences $a_1, \dots a_n \in A$

$$\left(\sum_{j=1}^{n} \|\varphi(a_j)\|^2\right)^{\frac{1}{2}} \leqslant C\|\varphi\|_{cb} \left\|\sum_{j=1}^{n} a_j^* a_j\right\|^{\frac{1}{2}}.$$

But since X is 2-concave,

$$\begin{aligned} \left(\sum_{j=1}^{n} \|\varphi(a_j)\|^2\right)^{\frac{1}{2}} &\leqslant C \left\|\sum_{j=1}^{n} \varphi(a_j)^* \varphi(a_j)\right\|^{\frac{1}{2}} \\ &\leqslant C\|\varphi\|_{cb} \left\|\sum_{j=1}^{n} a_j^* a_j\right\|^{\frac{1}{2}} \end{aligned}$$

as required. □

Remark

In the above proposition, we do not need that φ is completely bounded, only that φ is bounded on columns; that is, that for all finite sequences $a_1, \dots a_n \in A$

$$\left\|\varphi_n \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\right\|^2 = \left\|\begin{pmatrix} \varphi(a_1) \\ \vdots \\ \varphi(a_n) \end{pmatrix}\right\|^2 = \left\|\sum_{j=1}^{n} \varphi(a_j)^* \varphi(a_j)\right\| \leqslant C \left\|\sum_{j=1}^{n} a_j^* a_j\right\| = C \left\|\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\right\|^2.$$

where C is a constant depending only on φ.

Clearly, if φ is completely bounded then it is bounded on columns.

We now demonstrate the necessity of 2-concavity of X.

Suppose that for a subspace X of a C*-algebra A there exists a constant C such that for all completely bounded maps $\varphi : X \to X$ and for all finite sequences $a_1, \ldots a_n \in X$

$$\left(\sum_{j=1}^{n} \|\varphi(a_j)\|^2\right)^{\frac{1}{2}} \leqslant C\|\varphi\|_{cb} \left\|\sum_{j=1}^{n} a_j^* a_j\right\|^{\frac{1}{2}}$$

Then in particular the identity on X satisfies the above inequality, so for any finite sequence $x_1, \ldots, x_n \in X$

$$\left(\sum_{j=1}^{n} \|x_j\|^2\right)^{\frac{1}{2}} \leqslant C \left\|\sum_{j=1}^{n} x_j^* x_j\right\|^{\frac{1}{2}}$$

Hence X must be 2-concave.

References

[AW] C. A. AKEMANN and S. WRIGHT

Compact actions on C*-algebras *Glasgow Math. J.* **21**(1980) 143–149

[A] J. C. ALEXANDER

Compact Banach algebras *Proc. London Math. Soc. (3)* **18**(1968)

[Ar] W. B. ARVESON

Subalgebras of C*-algebras *Acta Math.* **123**(1969) 144–224

[BL] J. BERGH and J. LÖFSTRÖM

Interpolation spaces, an introduction Springer-Verlag (1976)

[B1] D. P. BLECHER

Geometry of the tensor product *Math. Proc. Cambridge Philos. Soc.* **104**(1988) 119–127

[B2] D. P. BLECHER

The standard dual of an operator space, preprint

[B3] D. P. BLECHER

Tensor products of operator spaces II, preprint(1990)

[BP] D. P. BLECHER and V. I. PAULSEN

Tensor products of operator spaces *J. Funct. Anal.*, to appear

[BS] D. P. BLECHER and R. R. SMITH

The dual of the Haagerup tensor product, preprint(1990)

[BD] F. F. BONSALL and J. DUNCAN

Complete normed algebras Springer-Verlag (1973)

[ChS] A. CHATTERJEE and A. M. SINCLAIR

An isometry from the Haagerup tensor product into completely bounded operators *J. Operator Theory*, to appear

[ChSm] A. CHATTERJEE and R. R. SMITH

The central Haagerup tensor product, preprint (1991)

[C] M. CHODA

Group factors of the Haagerup type *Proc. Japan Acad.* **59**(1983) 174–177

[CES] E. CHRISTENSEN, E. G. EFFROS and A. M. SINCLAIR

Completely bounded maps and C*-algebraic cohomology *Invent. Math.* **90**(1987) 279–296

[CS1] E. CHRISTENSEN and A. M. SINCLAIR

Representations of completely bounded multilinear operators *J. Funct. Anal.* **72**(1987) 151–181

[CS2] E. CHRISTENSEN and A. M. SINCLAIR

A survey of completely bounded operators *Bull. London Math. Soc.* **21**(1989) 417–448

[CS3] E. CHRISTENSEN and A. M. SINCLAIR

On the Hochschild cohomology for von Neumann algebras, preprint

[Da] A. M. DAVIE

The approximation problem for Banach spaces *Bull. London Math. Soc.* **5**(1973) 261–266

[D] J. DIXMIER

Les algèbres d'opérateurs dans l'espace Hilbertien Gauthier-Villars (1969)

[DS] N. DUNFORD and J. T. SCHWARTZ

Linear Operators, part I Interscience (1958)

[E1] E. G. EFFROS

Advances in quantized functional analysis *Proc. ICM Berkeley 1986* (Amer. Math. Soc., 1987) 906–916

[D2] J. DIXMIER
C*-algebras

[E2] E. G. EFFROS

On multilinear completely bounded module maps *Contemporary Math.* **62**(1987) 479–501

[EK] E. G. EFFROS and A. KISHIMOTO

Module maps and Hochschild-Johnson cohomology *Indiana Math J.* **36**(1987) 257–276

[ER1] E. G. EFFROS and Z-J. RUAN

On approximation properties for operator spaces *Int. J. of Math.* **1**(1990) 163–187

[ER2] E. G. EFFROS and Z-J. RUAN

A new approach to operator spaces, preprint

[ER3] E. G. EFFROS and Z-J. RUAN

Self-duality for the Haagerup tensor product and Hilbert space factorisation, *J. Funct. Anal.*, to appear

[ER4] E. G. EFFROS and Z-J. RUAN

The Grothendieck-Pietsch and Dvoretzky-Rogers theorems for operator spaces, preprint(1991)

[En] P. ENFLO

A counterexample to the approximation property in Banach spaces *Acta Math.* **130**(1973) 309–317

[G1] A. GROTHENDIECK

Produits tensoriels topologiques et espaces nucléaires *Mem. Amer. Math. Soc. No.16*(1955)

[G2] A. GROTHENDIECK

Résumé de la théorie métrique des produits tensoriels topologiques *Bol. Soc. Mat. São-Paulo* **8**(1956) 1–79

[H1] U. Haagerup
Solution of the similarity problem for cyclic representations of C*-algebras *Ann. of Math.* **118**(1983) 215–240

[H2] U. Haagerup
A new proof of the equivalence of injectivity and hyperfiniteness for factors on a separable Hilbert space *J. Funct. Anal.* **62**(1985) 160–201

[H3] U. Haagerup
The Grothendieck inequality for bilinear forms on C*-algebras *Adv. in Math.* **56**(1985) 93–116

[H4] U. Haagerup
Decomposition of completely bounded maps on operator algebras, unpublished manuscript

[Ha] P. R. Halmos
A Hilbert space problem book Springer-Verlag (1982)

[J] G. J. O. Jameson
Summing and nuclear norms in Banach space theory London Math. Soc. Student Texts 8, (CUP, 1987)

[K] R. V. Kadison
On the orthogonalization of operator representations *Amer. J. Math.* **77**(1955) 600–620

[KR] R. V. Kadison and J. R. Ringrose
Fundamentals of the theory of operator algebras Vols.1, 2 Academic Press (1986)

[Ka] J. P. Kahane
Some random series of functions (2nd ed.) Cambridge Studies in Advanced Mathematics No.5 (1985)

[KS] S. Kaijser and A. M. Sinclair
Projective tensor products of C*-algebras *Math. Scand* **55**(1984) 161–187

[Kw] S. KWAPIEŃ
Isomorphic characterisations of inner product spaces by orthogonal series with vector coefficients *Studia Math.* **44**(1972) 583–595

[LT] J. LINDESTRAUSS and L. TZAFRIRI
Classical Banach spaces, vols.I,II Springer-Verlag (1974)

[LPP] F. LUST-PIQUARD and G. PISIER
Non-commutative Khintchine and Paley inequalities, preprint(1990)

[M1] M. MATHIEU
Elementary operators on prime C*-algebras II *Glasgow Math. J.* **30**(1988) 275–284

[M2] M. MATHIEU
Compact and weakly compact multiplications on C*-algebras *Ann. Acad. Sci. Fenn. Ser.A I Math.* **14**(1989) 57–62

[MP] B. MAUREY and G. PISIER
Séries de variables aléatoires vectorielles indépendentes et propriétés géométriques des espaces de Banach *Studia Math.* **58**(1976) 45–90

[Pa] V. I. PAULSEN
Completely bounded maps and dilations Pitman Research Notes in Math. 146 (Longmans, 1986)

[PS] V. I. PAULSEN and R. R. SMITH
Multilinear maps and tensor norms on operator algebras *J. Funct. Anal.* **73**(1987) 258–276

[Pe] G. K. PEDERSEN
C^*-algebras and their automorphism groups Academic Press (1979)

[P] A. PIETSCH
Absolut p-summierende Abbildungen in normierten Raümen *Studia Math.* **28**(1967) 333–353

[Pi1] G. Pisier

Grothendieck's theorem for non-commutative C*-algebras with an appendix on Grothendieck's constants *J. Funct. Anal.* **29**(1978) 397–415

[Pi2] G. Pisier

Un théorème sur les opérateurs entre espaces Banach qui factorisent par un espace de Hilbert *Ann. École Norm. Sup.* **13**(1980) 23–43

[Pi3] G. Pisier

Counterexamples to a conjecture of Grothendieck *Acta Math.* **151**(1983) 181–203

[Pi4] G. Pisier

Factorisation of linear operators an geometry of Banach spaces CBMS Series No.60 (Amer. Math. Soc. Publ., 1986)

[R] C. E. Rickart

Banach algebras Van Nostrand Rheinhold (1960)

[Ri] J. R. Ringrose

Compact non-self-adjoint operators Math. Studies 35 (Van Nostrand Rheinhold, 1971)

[Ro] J. E. Roberts

Cross-products of von Neumann algebras by group duals *Symp. Math.* **20**(1976) 335–363

[Rob] A. G. Robertson

Injective matricial Hilbert spaces *Math. Proc. Cambridge Philos. Soc.* (1991)

[Ru] Z-J. Ruan

Subspaces of C*-algebras *J. Funct. Anal.* **76**(1988) 217–230

[Sa] S. Sakai

C-algebras and W* -algebras* Springer-Verlag (1971)

[S] R. R. SMITH

Completely bounded module maps and the Haagerup tensor product *J. Funct. Anal.*, to appear

[St] W. F. STINESPRING

Positive functions on C*-algebras *Proc. Amer. Math. Soc.* **6**(1955) 211–216

[Sz] A. SZANKOWSKI

$B(H)$ does not have the approximation property *Acta Math.* **147**(1981) 89–108

[T] M. TAKESAKI

Theory of operator algebras I Springer-Verlag (1979)

[Ta] A. E. TAYLOR

Introduction to functional analysis Wiley (1958)

[TJ] N. TOMCZAK-JAEGERMANN

The moduli of smoothness and convexity and the Rademacher averages of the trace classes $S_p(1 \leq p < \infty)$ *Studia Math.* **50**(1974) 163–182

[To] J. TOMIYAMA

On the transpose map of matrix algebras *Proc. Amer. Math. Soc.* **88**(1983) 635–638

[U] H. UMEGAKI

Conditional expectations in an operator algebra *I. Tôhoku Math. J.* **6**(1954) 177–181

[V1] K. VALA

On compact sets of compact operators *Ann. Acad. Sci. Fenn. Ser. AI Math.* **351**(1964)

[V2] K. VALA

Sur les éléments compacts d'une algèbre normée *Ann. Acad. Sci. Fenn. Ser. AI Math.* **407**(1967)

[W1] G. Wittstock

Ein operatorwertiger Hahn-Banach Satz *J. Funct. Anal.* **40**(1981) 127–150

[W2] G. Wittstock

On matrix order and convexity *Functional analysis, surveys and recent results* Math. Studies 90 (North-Holland, 1984) pp175–188